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Scaling to Preserve iISS Dissipation Inequalities for Nonlinear Systems

Hiroshi Ito
Dept. of Systems Design and Informatics
Kyushu Institute of Technology, Japan

Joint work with

Christopher M. Kellett
School of Electrical Engineering and Computer Science
The University of Newcastle, Australia

ISS and iISS

$\Sigma \; : \; \dot{x}(t) = f(x(t), r(t))$

ISS (Input-to-State Stable) ...Sontag(1989)

$\exists\beta \in \mathcal{KL}$: global asymptotic stability
$\exists\gamma \in \mathcal{K}$: nonlinear gain function
s.t. for any $x(0) \in \mathbb{R}^N$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{\tau\in[0,t]} |r(\tau)|\right), \quad \forall t \in [0, \infty)$$

iISS (integral Input-to-State Stable) ...Sontag(1998)

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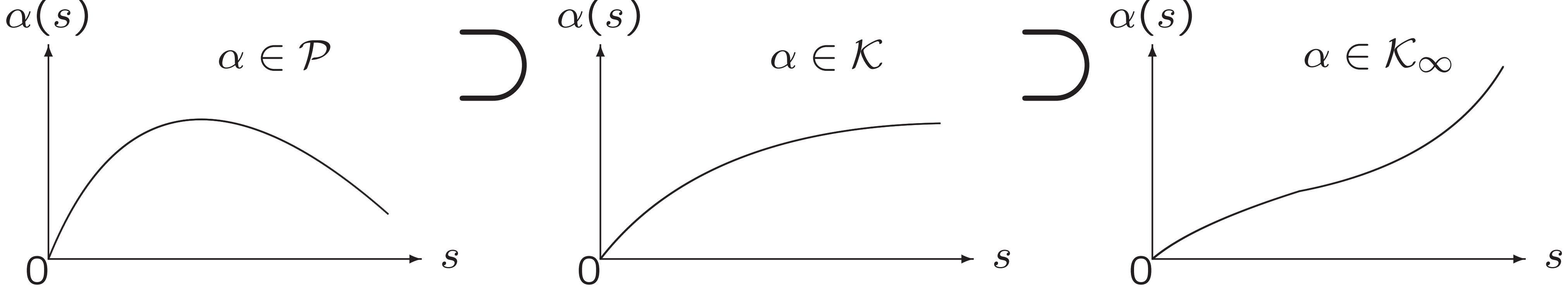
$\exists\eta \in \mathcal{K}$, $\exists\mu \in \mathcal{K}$
s.t. for any $x(0) \in \mathbb{R}^N$,

$$|x(t)| \leq \beta(|x(0)|, t) + \eta\left(\int_0^t \mu(|r(\tau)|)d\tau\right), \ \forall t \in [0, \infty)$$

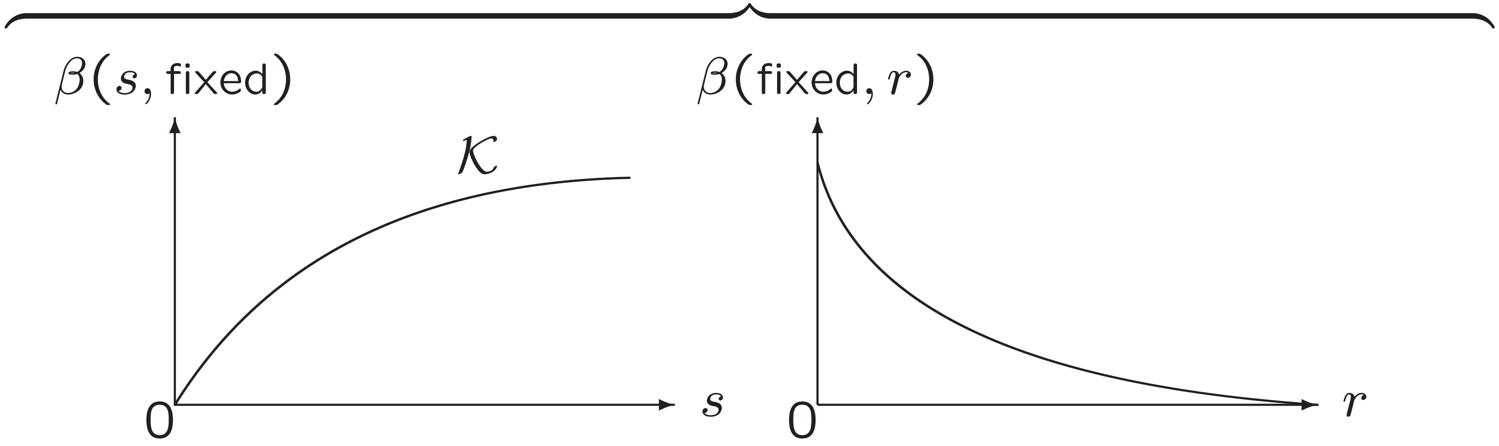
Positive Definite($\mathcal{P}$), Class $\mathcal{K}$, Class $\mathcal{K}_\infty$, Class $\mathcal{KL}$

invertible on $[0, \infty)$



$$\lim_{s \to \infty} \alpha(s) = \infty$$

$$\beta \in \mathcal{KL}$$



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↓

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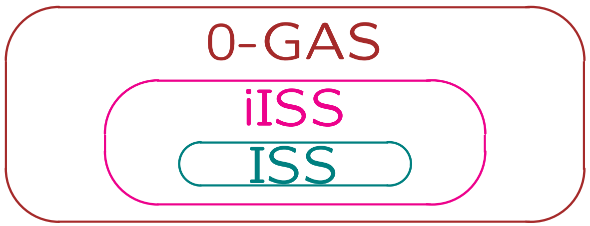
$$\underbrace{|x(t)| \le \beta(|x(0)|, t)}_{\text{0-GAS}} + \eta \underbrace{\left(\int_0^t \mu(|r(\tau)|) d\tau\right)}_{\text{can be unbounded}}, \quad \forall t \in [0, \infty)$$

↘ larger upper bound



Lyapunov Characterization of ISS and iISS

$$\frac{\partial V}{\partial x} f(x,r) \le -\alpha(V(x)) + \sigma(|r|) \ , \quad \exists \sigma \in \mathcal{K}, V$$ Sontag&Wang'95, Angeli et al.'00

$\exists \alpha \in \mathcal{P}$ $\Leftrightarrow$ $\dot{x} = f$ is iISS.

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Examples:

$\alpha(s) = s$	, $\sigma(s) = s$	$\cdots$ ISS	, Gain $= \alpha^{-1} \circ \sigma \in \mathcal{K}$
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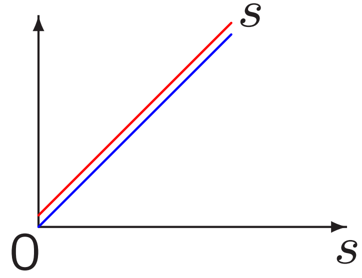

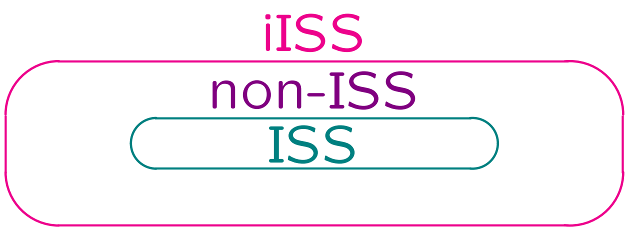

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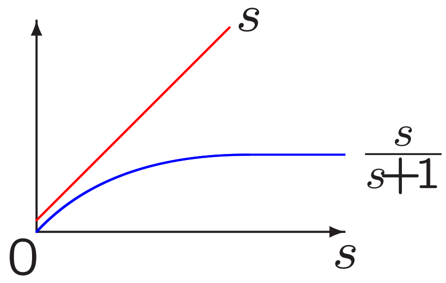

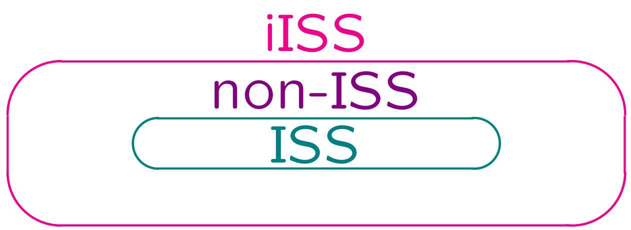

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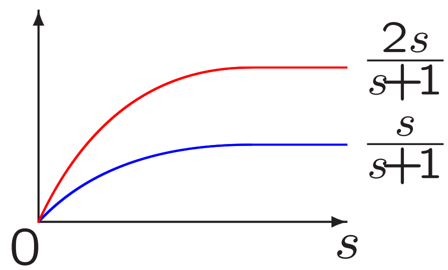

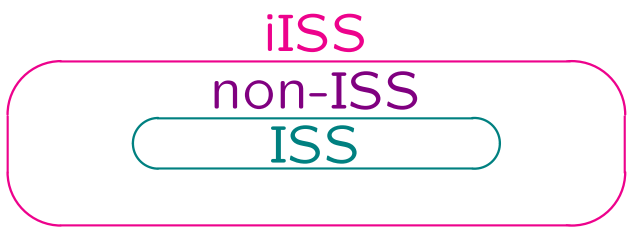

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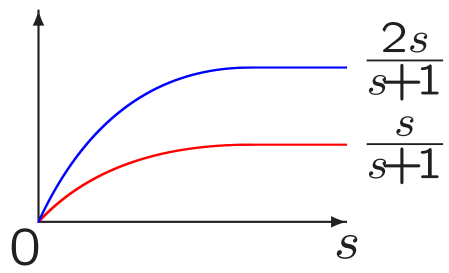

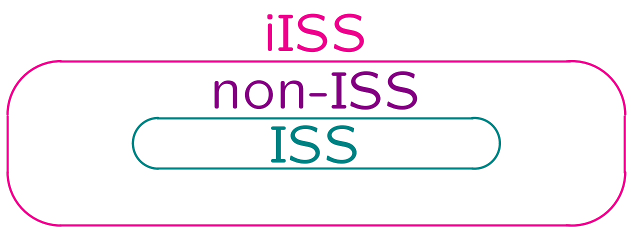

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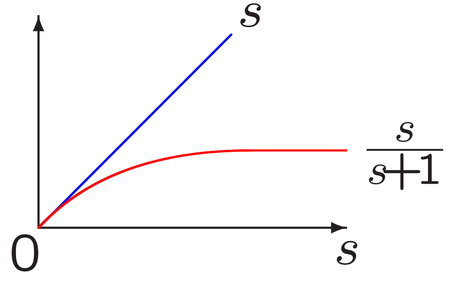
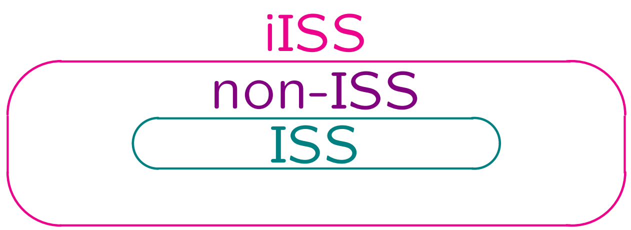
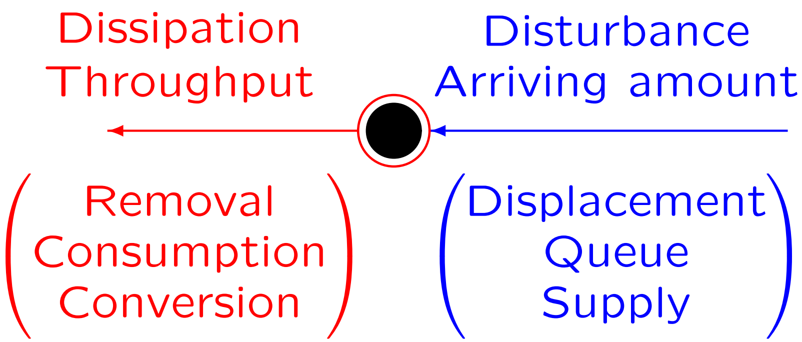
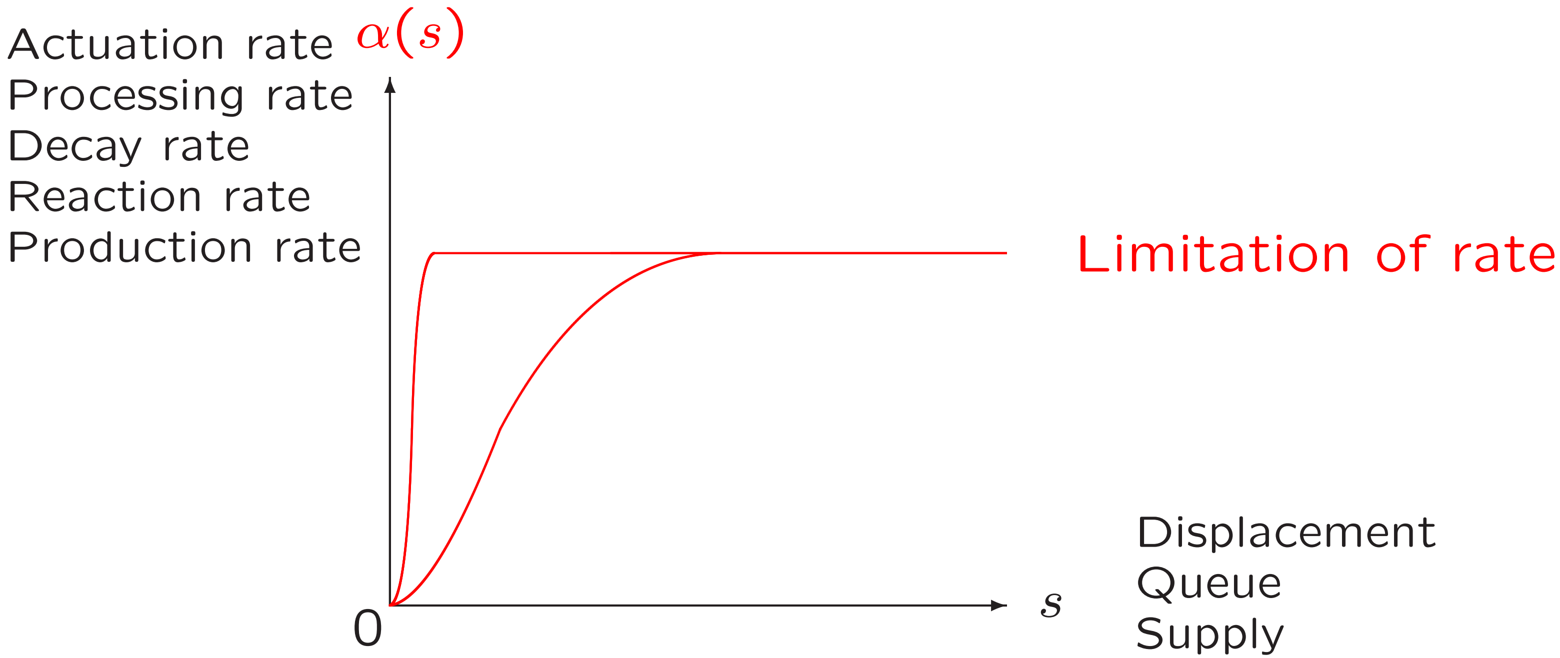




Examples Is Everywhere

Examples arise in
- Tracking
- Data processing
- Traffic processing
- Biochemical processes
- Production lines
- . . .





Actuator limitations, Michaelis-Menten (Monod) kinetics, Limited capacity/resources

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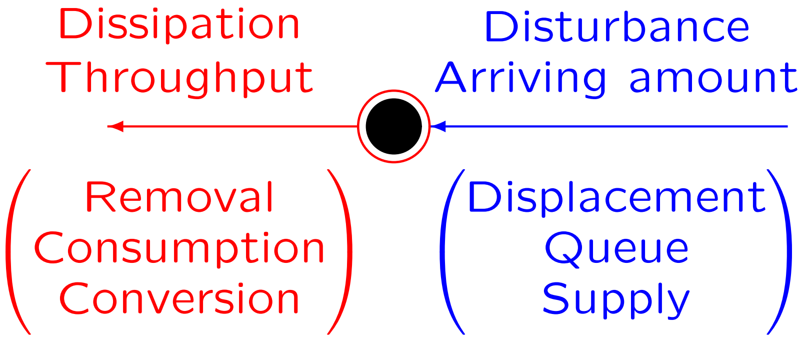
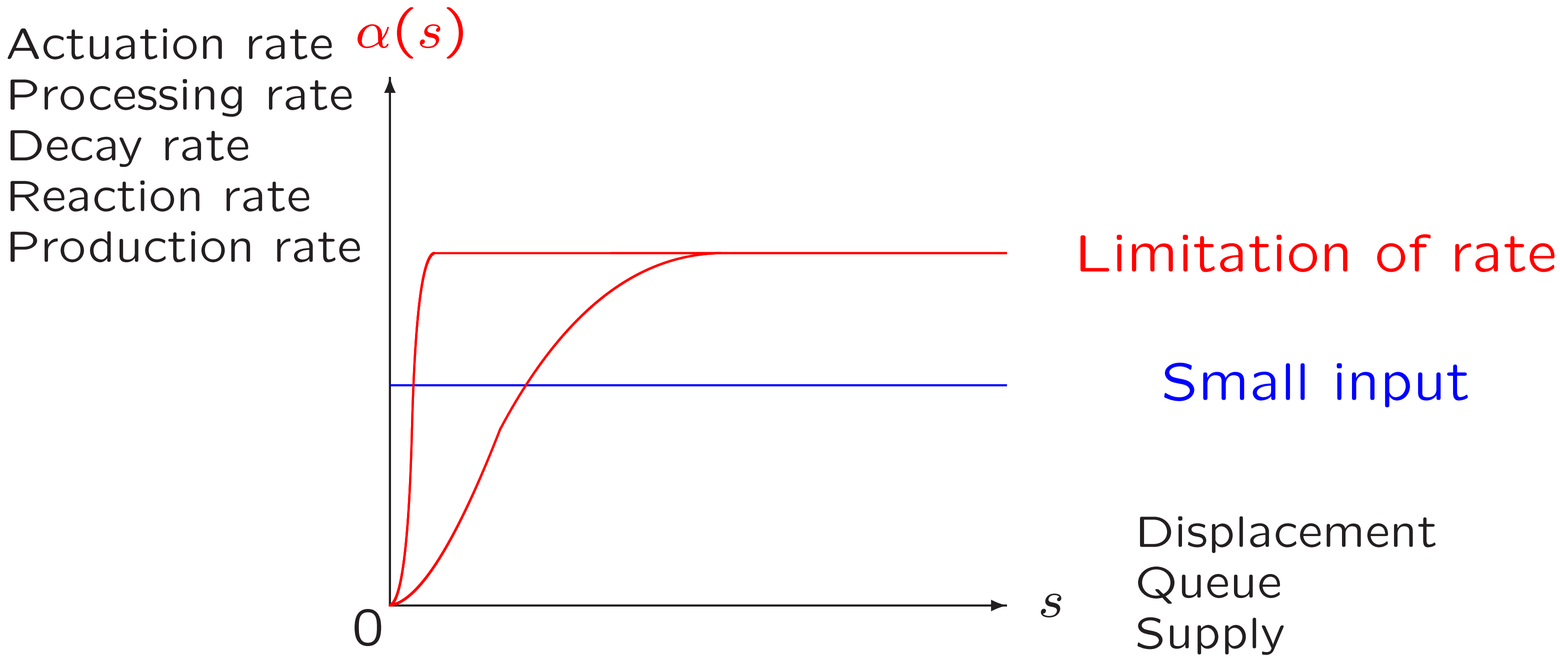
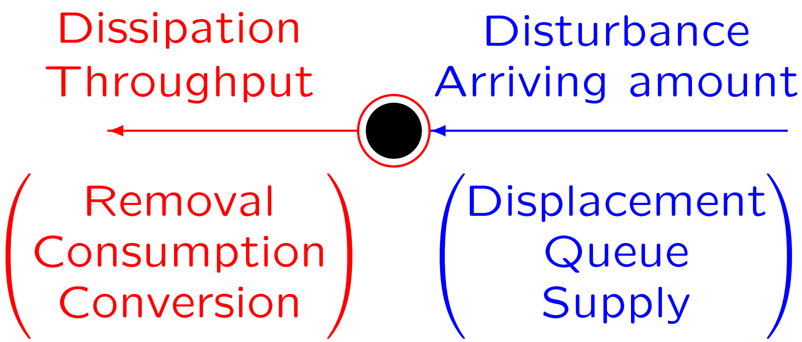


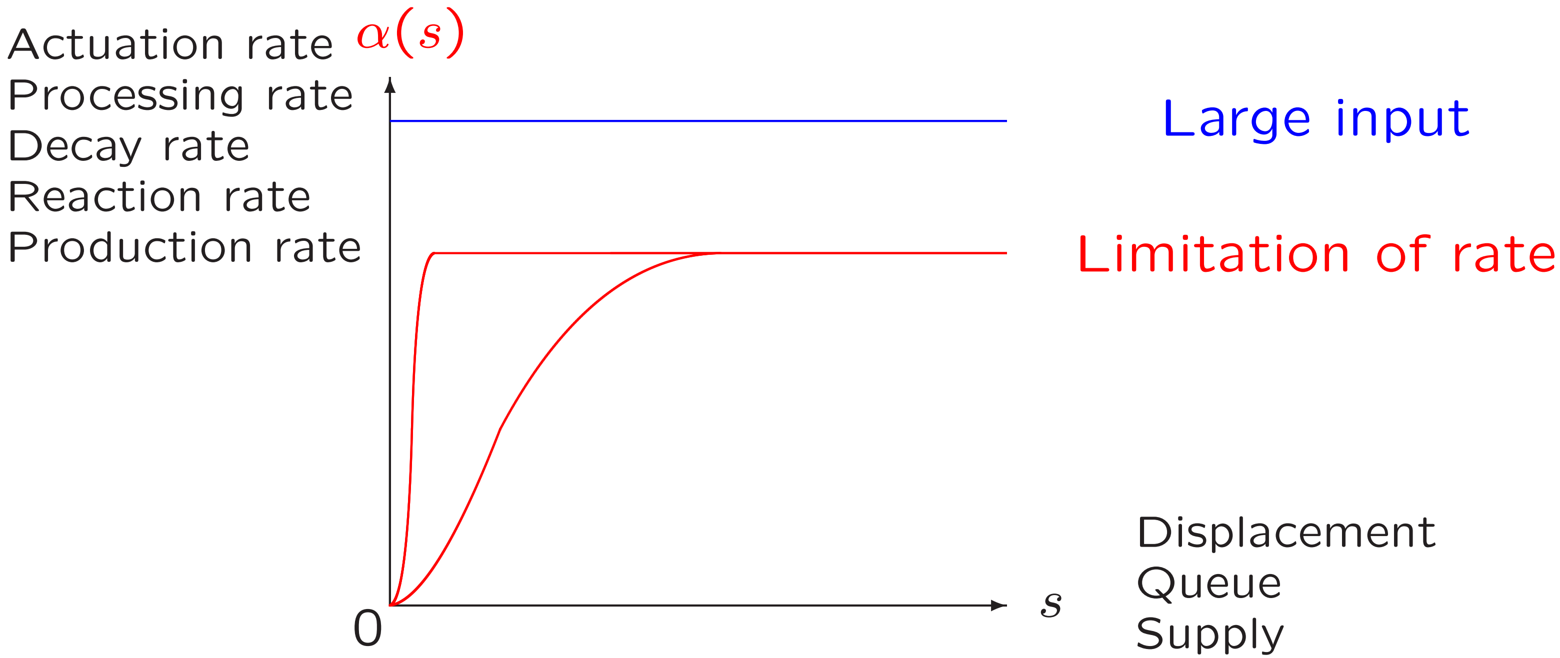


The internal quantity remains bounded.

Examples Is Everywhere

Examples arise in { Tracking, Data processing, Traffic processing, Biochemical processes, Production lines, . . .





The internal quantity increases unboundedly.

Two Central Tools in Lyapunov-Based Analysis and Design

$\dot{x}(t) = f(x(t), w(t))$

$V(x)$: A Lyapunov function in the presence of the external signal w

Scaling

$$\frac{\partial V}{\partial x}(x) f(x,w) = R(x,w)$$

$$W = \mu(V) \Downarrow \quad \mu \in \mathcal{S} := \{\mathcal{K}_\infty \cap \mathcal{C}^1, \mu'(s) > 0,\ \forall s \in (0,\infty)\}$$

$$\frac{\partial W}{\partial x}(x) f(x,w) = \mu'(V(x)) R(x,w)$$

Dissipation inequality

$$\frac{\partial V}{\partial x}(x) f(x,w) \leq -\alpha(V(x)) + \sigma(|w|) \cdots \begin{cases} \text{ISS}, & \text{if } \alpha \text{ dominates over } \sigma \\ \text{iISS}, & \text{otherwise.} \end{cases}$$

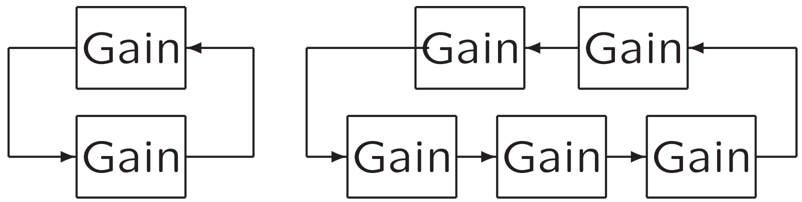
Questions to be answered

- Under what condition does an iISS (resp. ISS) dissipation inequality remains iISS (resp. ISS)?
- Limitations, Explicit (less conservative) formulas, Benefits?

Why Important? Useful?

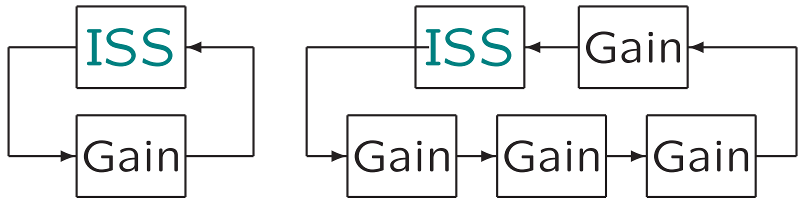
Preservation of iISS

Small-gain arguments emerge.

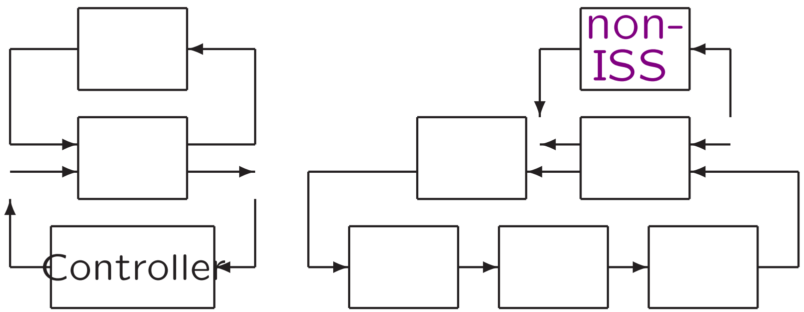


Preservation of ISS

At least one system in a loop is necessarily ISS to guarantee stability of the loop.



Explicit formulas of scalings



Having a Lyapunov func. is useful for

controller design;

allowing other component systems to be weaker than ISS.

Flexibility in Lyapunov functions can lead to a better controller.

Observations

- Scalings do not always preserve iISS of dissipation inequalities:

$$\frac{\partial V}{\partial x}(x)f(x,w) \le -\frac{2V}{1+V} + w^2$$

$$W=\mu(V)=V^2 \quad \Downarrow$$

$$\frac{\partial V}{\partial x}(x)f(x,w) \le -2V\frac{2V}{1+V} + 2Vw^2 \not\le -\hat{\alpha}(W)+\hat{\sigma}(|w|)$$

Large V prohibits "$\le \hat{\sigma}(|w|)$".

- For interconnection, better scalings give a simpler Lyapunov function:

$$\dot{x}_1 = -\frac{x_1}{x_1^2+2} + \frac{5x_1x_2^2}{4(x_1^2+2)(x_2^2+3)}, \qquad \dot{x}_2 = -x_2 - \frac{8x_1}{7|x_1|+9}$$

With $V_1 = x_1^2/2$ and $V_2 = x_2^2/2$, the iISS small-gain theorem (Ito&Jiang '09, Ito '13) verifies GAS of $(x_1,x_2)=0$ through

$$V(x) = \int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{26}\left[\frac{5s}{7s+6}\right]^{27} ds + \int_0^{V_2(x_2)} s^{26}\left[\frac{5s}{4s+6}\right]^{27} ds,$$

but numerical computation hints that 26,27 can be made smaller.

iISS Preservation: A Necessary Cond. and A Coarse Estimate

Given $\left\{ \begin{array}{l} V : \mathbb{R}^N \to \mathbb{R}_+,\ \mathcal{C}^1, \text{positive definite, radially unbounded} \\ \alpha \in \mathcal{P},\ \sigma \in \mathcal{K} \end{array} \right\}$.

Assume $\mu \in \mathcal{S}$ in $W(x) = \mu(V(x))$.

Theorem 1 (A necessary condition in the non-ISS case) Assuming that $\lim_{s\to\infty} \alpha(s)$ exists.

Suppose that $\underbrace{\lim_{s\to\infty} \alpha(s) < \lim_{s\to\infty} \sigma(s)}_{\text{iISS, but not ISS}}$. Then

$\exists \hat{\alpha} \in \mathcal{P},\ \hat{\sigma} \in \mathcal{K}$ s.t.

$$\mu'(V(x))\left[-\alpha(V(x)) + \sigma(|w|)\right] \le -\hat{\alpha}(W(x)) + \hat{\sigma}(|w|),\ \forall x \in \mathbb{R}^N, w \in \mathbb{R}^M \quad \cdots \ (\textit{TransformedDI})$$

$$\Downarrow$$

$$\limsup_{s\to\infty} \mu'(s) < \infty.$$

Note:

$$\frac{\partial V}{\partial x} f(x,w) \le -\alpha(V(x)) + \sigma(|w|) \ \Rightarrow\ \frac{\partial W}{\partial x} f(x,w) \le -\mu'(V(x))\left[-\alpha(V(x)) + \sigma(|w|)\right]$$

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Proposition 1 (A coarse estimate)

$$\limsup_{s\to\infty} \mu'(s) < \infty$$

$$\Downarrow$$

$$\hat{\alpha} = [\mu' \circ \mu^{-1}][\alpha \circ \mu^{-1}] \in \mathcal{P},\ \hat{\sigma} = \sup_{l \in \mathbb{R}_+} \mu'(l)\,\sigma \in \mathcal{K} \text{ satisfy } (\textit{TransformedDI}).$$

iISS/ISS Preservation: Division Approach

$$\mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)] \le -\mu'(V(x))\alpha(V(x))+\mu'(V(x))\sigma(|w|))$$
$$\le -\widehat{\alpha}_*(W(x))+\widehat{\sigma}_*(|w|) \quad \swarrow ?\cdots \quad (\mathit{Transformed}\,DI)$$

Division technique (ISS$\cdots$Sontag&Teel'95, iISS$\cdots$Ito'06):

$$\mu'\sigma=\zeta(\alpha)\sigma \le \left\{\begin{array}{ll}\zeta(\alpha)\frac{1}{\tau}\alpha, & \text{for } \alpha\ge\tau\sigma\\ \zeta(\tau\sigma)\sigma, & \text{for } \alpha<\tau\sigma\end{array}\right\} \le \frac{1}{\tau}\zeta(\alpha)\alpha+\zeta(\tau\sigma)\sigma$$

τ : Positive constant

Theorem 2 Assume that $\mu\in\mathcal{S}$ and $\alpha\in\mathcal{K}$ satisfy

$\mu'(s)$ is non-decreasing;

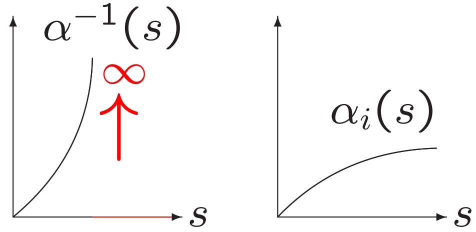
$\lim_{s\to\infty}\alpha(s)<\lim_{s\to\infty}\sigma(s) \;\Rightarrow\; \lim_{s\to\infty}\mu'(s)<\infty.$

Then there exists $\tau>1$ such that ($\mathit{Transformed}\,DI$) holds with

$$\widehat{\alpha}_D=[\mu'\circ\mu^{-1}]\left[\left(1-\frac{1}{\tau}\right)\alpha\circ\mu^{-1}\right]\in\mathcal{P}\ \cdots\ \text{iISS preserved}$$

$$\widehat{\sigma}_D=[\mu'\circ\alpha^{\ominus}\circ\tau\sigma]\sigma\in\mathcal{K}\ .$$

where $\alpha^{\ominus}(s)=\left\{\begin{array}{ll}\alpha^{-1}(s) & \text{if } \lim_{\tau\to\infty}\alpha(\tau)>s\\ \infty & \text{otherwise}\end{array}\right.$



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$\mu'(s)$ is non-decreasing;

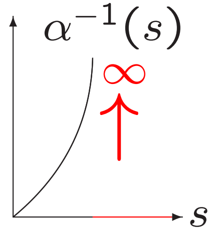
$\lim_{s\to\infty} \alpha(s) < \infty \quad \Rightarrow \quad \lim_{s\to\infty} \mu'(s) < \infty.$

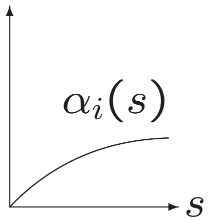
Then for any $\tau > 1$, $(\mathit{Transformed\,DI})$ holds with

$$\hat{\alpha}_D=[\mu'\circ\mu^{-1}]\left[\left(1-\frac{1}{\tau}\right)\alpha\circ\mu^{-1}\right] \in \mathcal{P} \cdots \text{ iISS preserved}$$

$$\hat{\sigma}_D=[\mu'\circ\alpha^{\ominus}\circ\tau\sigma]\sigma \in \mathcal{K}\ .$$

where $\alpha^{\ominus}(s)=\begin{cases} \alpha^{-1}(s) & \text{if } \lim_{\tau\to\infty}\alpha(\tau)>s \\ \infty & \text{otherwise} \end{cases}$





iISS/ISS Preservation: Division Approach

$$\mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)] \le -\mu'(V(x))\alpha(V(x)) + \mu'(V(x))\sigma(|w|))$$
$$\le -\widehat{\alpha}_*(W(x))+\widehat{\sigma}_*(|w|) \quad \swarrow ? \cdots \quad (\textit{TransformedDI})$$

Division technique (ISS$\cdots$Sontag&Teel'95, iISS$\cdots$Ito'06):

$$\mu'\sigma = \zeta(\alpha)\sigma \le \left\{ \begin{array}{ll} \zeta(\alpha)\frac{1}{\tau}\alpha, & \text{for } \alpha \ge \tau\sigma \\ \zeta(\tau\sigma)\sigma, & \text{for } \alpha < \tau\sigma \end{array} \right\} \le \frac{1}{\tau}\zeta(\alpha)\alpha + \zeta(\tau\sigma)\sigma$$

τ : Positive constant

Theorem 2 Assume that $\mu \in \mathcal{S}$ and $\alpha \in \mathcal{K}$ satisfy

$\mu'(s)$ is non-decreasing;

$\lim_{s\to\infty} \alpha(s) < \infty \quad \Rightarrow \quad \lim_{s\to\infty} \mu'(s) < \infty.$

Then for any $\tau > 1$, (*TransformedDI*) holds with

$$\widehat{\alpha}_D = [\mu' \circ \mu^{-1}]\left[\left(1-\frac{1}{\tau}\right)\alpha \circ \mu^{-1}\right] \in \mathcal{P} \cdots \text{ iISS preserved}$$

$$\widehat{\sigma}_D = [\mu' \circ \alpha^{\ominus} \circ \tau\sigma]\sigma \in \mathcal{K}\ . \cdots \text{ Idea of less conservative estimate}$$

Furthermore, if $\lim_{s\to\infty} \alpha(s)=\infty$ or $(\tau-1)\lim_{s\to\infty} \mu'(s) \ge \lim_{s\to\infty} \mu'\circ\alpha^{\ominus}\circ\tau\sigma(s)$,

$$\left.\begin{array}{lll} \lim_{s\to\infty} \alpha(s)=\infty & \Rightarrow & \lim_{s\to\infty} \widehat{\alpha}_D(s)=\infty \\ \lim_{s\to\infty} \alpha(s) > \lim_{s\to\infty} \sigma(s) & \Rightarrow & \lim_{s\to\infty} \widehat{\alpha}_D(s) > \lim_{s\to\infty} \widehat{\sigma}_D(s) \end{array}\right. \cdots \text{ ISS preserved}$$

iISS/ISS Preservation: Division Approach

$$\mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)] \le -\mu'(V(x))\alpha(V(x)) + \mu'(V(x))\sigma(|w|))$$
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Furthermore, if $\lim_{s\to\infty} \alpha(s) = \infty$ or $\underbrace{(\tau-1)\lim_{s\to\infty}\mu'(s) \ge \lim_{s\to\infty}\mu'\circ\alpha^{\ominus}\circ\tau\sigma(s)}$,

always achievable by $\mu'(s) = \alpha(s)^{\varphi}\beta(s)$,

($\beta : \mathbb{R}_+ \to \mathbb{R}_+$ any non-decreasing)

ISS Preservation: Legendre-Fenchel Transform Approach

$$\mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)] \le -\mu'(V(x))\alpha(V(x))+\mu'(V(x))\sigma(|w|))$$
$$\le -\hat{\alpha}_*(W(x))+\hat{\sigma}_*(|w|) \quad \swarrow ? \cdots \quad (\mathit{TransformedDI})$$

Legendre-Fenchel approach (Praly&Jiang'93):

$\mu'\sigma \le \kappa(\mu') + \ell\kappa(\sigma)$ for any $\kappa, \kappa' \in \mathcal{K}_\infty$

LF transform: $\ell\kappa(s) := \int_0^s (\kappa')^{-1}(l)dl = s(\kappa')^{-1}(s) - \kappa\circ(\kappa')^{-1}(s)$

Theorem 3 Assume that $\mu \in \mathcal{S}$ and $\alpha \in \mathcal{K}$ satisfy

$\mu'(s)$ is non-decreasing;

$\lim_{s\to\infty}\alpha(s) < \infty \ \Rightarrow\ \lim_{s\to\infty}\mu'(s) < \infty$.

Then (*TransformedDI*) holds with

$$\hat{\alpha}_L = [\mu'\circ\mu^{-1}]\left[\alpha\circ\mu^{-1}\right] - \kappa\circ\mu'\circ\mu^{-1} \quad : \mathbb{R}_+ \to \mathbb{R}$$

$$\hat{\sigma}_L = \min\left\{\ell\kappa\circ\sigma,\ \lim_{s\to\infty}\mu'(l)\sigma\right\} \in \mathcal{K}\ .$$

Remark: If $\hat{\alpha}_L \in \mathcal{P}$, then iISS is preserved.

If $\lim_{s\to\infty}\hat{\alpha}_L(s)=\infty$ or $\lim_{s\to\infty}\hat{\alpha}_L(s)>\lim_{s\to\infty}\hat{\sigma}_L(s)$, then ISS is preserved.

ISS Preservation: Legendre-Fenchel Transform Approach

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Then (*TransformedDI*) holds with

$$\widehat{\alpha}_L = [\mu'\circ\mu^{-1}]\left[\alpha\circ\mu^{-1}\right] - \kappa\circ\mu'\circ\mu^{-1} \quad : \mathbb{R}_+ \to \mathbb{R}$$

$$\widehat{\sigma}_L = \min\left\{\ell\kappa\circ\sigma,\ \lim_{s\to\infty}\mu'(l)\sigma\right\} \in \mathcal{K} \ .$$

How to find such $\kappa,\kappa'\in\mathcal{K}_\infty$?

Remark: If $\widehat{\alpha}_L\in\mathcal{P}$, then iISS is preserved.

If $\lim_{s\to\infty}\widehat{\alpha}_L(s)=\infty$ or $\lim_{s\to\infty}\widehat{\alpha}_L(s)>\lim_{s\to\infty}\widehat{\sigma}_L(s)$, then ISS is preserved.

Relation between the Division and the LF Approaches

$$\mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)] \leq -\mu'(V(x))\alpha(V(x)) + \mu'(V(x))\sigma(|w|))$$
$$\leq -\widehat{\alpha}_*(W(x)) + \widehat{\sigma}_*(|w|) \quad \swarrow ? \cdots \quad (\textit{Transformed}\,DI)$$

Theorem 4 Assume that $\mu \in \mathcal{S}$ and $\alpha \in \mathcal{K}$ satisfy

$$\mu'(s) \text{ is strictly increasing;}$$
$$\lim_{s\to\infty} \alpha(s) < \infty \;\Leftrightarrow\; \lim_{s\to\infty} \mu'(s) < \infty;$$

Then for any $\tau > 1$, there exists $\kappa \in \mathcal{K}_\infty$ such that $\kappa' \in \mathcal{K}_\infty$ and

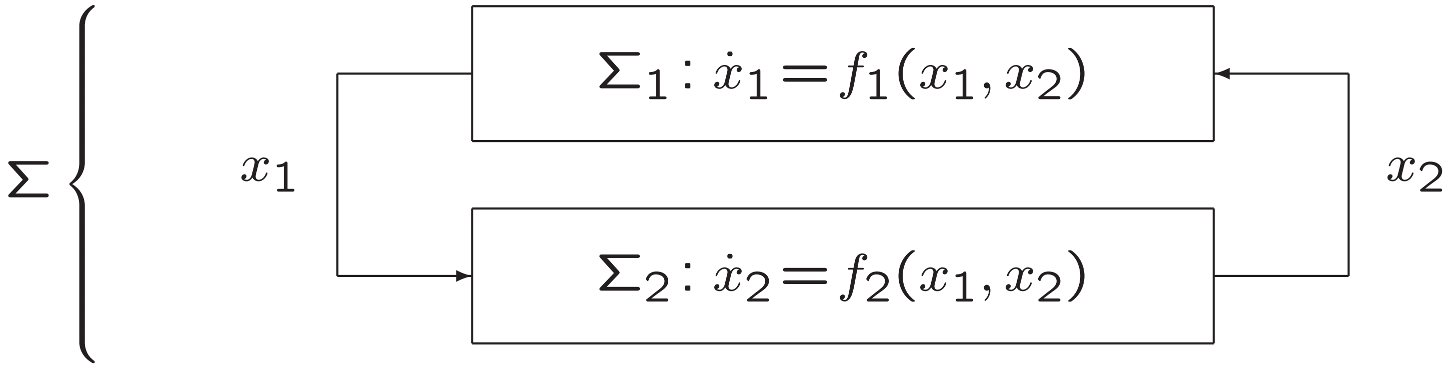
$$\widehat{\alpha}_L(s) \geq \widehat{\alpha}_D(s), \qquad \forall s \in \mathbb{R}_+$$
$$\widehat{\sigma}_L(s) \leq \widehat{\sigma}_D(s), \qquad \forall s \in \mathbb{R}_+$$

LF approach   Division approach

Remark:

- In preserving iISS/ISS, the LF approach gives tighter estimates.
- Two explicit choices: $\kappa(s) = \frac{1}{\tau} s[\alpha \circ \lambda^{\ominus}(s)]$, $\kappa'(s) = \frac{1}{\tau}\alpha \circ \lambda^{\ominus}(s)$

Interconnection of Two iISS Systems



$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Assumption

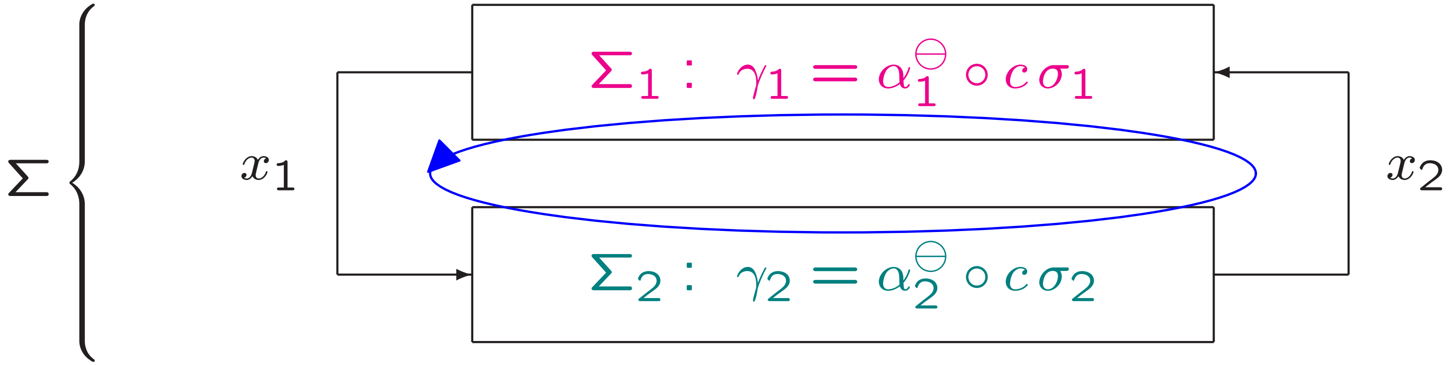
There exist $\left\{ \begin{array}{c} \text{positive definite, radially unbounded, } \mathbf{C}^1 \\ \text{functions } V_i : \mathbb{R}^{N_i} \to \mathbb{R}, \quad i = 1, 2, \\ \alpha_i \in \mathcal{K}, \ \sigma_i \in \mathcal{K} \end{array} \right\}$ such that

$$\frac{\partial V_1}{\partial x_1} f_1 \leq -\alpha_1(V_1(x_1)) + \sigma_1(V_2(x_2))$$

$$\frac{\partial V_2}{\partial x_2} f_2 \leq -\alpha_2(V_2(x_2)) + \sigma_2(V_1(x_1)) \qquad \forall x \in \mathbb{R}^N$$

i.e., Σ_i is iISS w.r.t input (x_{3-i}, r_i)

iISS/ISS Preservation Leads to Small-Gain Theorem



Theorem (Ito&Jiang '09) GAS$\cdots x = 0$ is globally asymptotically stable.

There exists $c > 1$ such that

$$\underbrace{\underbrace{\gamma_1}_{\Sigma_1} \circ \underbrace{\gamma_2(s)}_{\Sigma_2}}_{\text{Loop gain}} \leq s, \quad \forall s \in [0, \infty) \quad \Longrightarrow \quad \Sigma \text{ is GAS}$$

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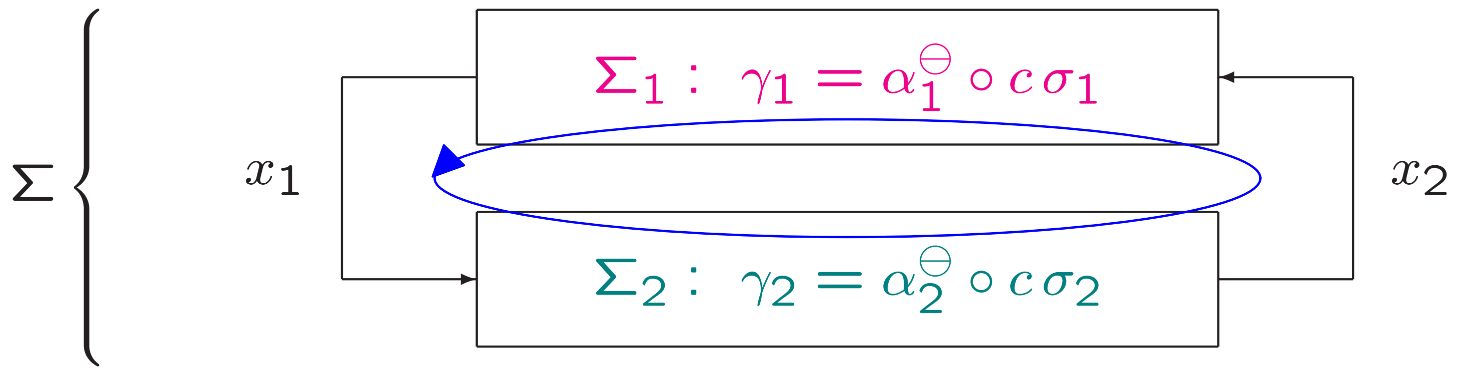

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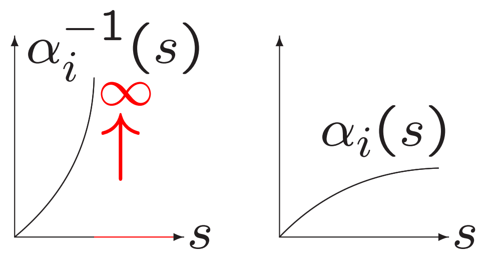
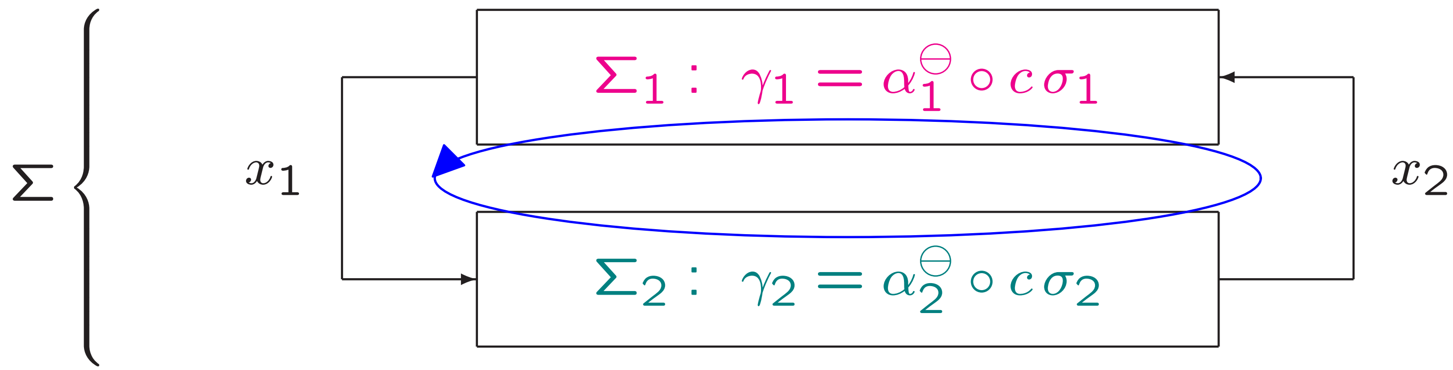
$$\underbrace{\underbrace{\gamma_1}_{\text{Infinite}} \circ \underbrace{\gamma_2(s)}_{\text{Bounded}}}_{\text{Loop gain}} \leq s, \quad \forall s \in [0, \infty) \implies \Sigma \text{ is GAS}$$

$$\text{where } \alpha_i^{\ominus}(s) = \begin{cases} \alpha_i^{-1}(s) & \text{if } \lim_{\tau \to \infty} \alpha_i(\tau) > s \\ \infty & \text{otherwise} \end{cases}$$



iISS/ISS Preservation Leads to Small-Gain Theorem



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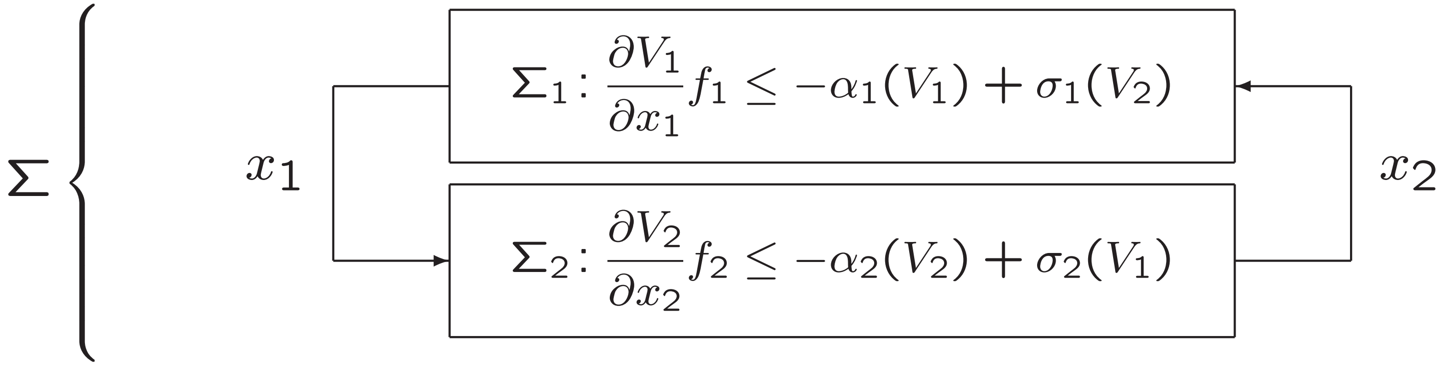

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There exists $c > 1$ such that

$$\underbrace{\underbrace{\gamma_1}_{\Sigma_1} \circ \underbrace{\gamma_2(s)}_{\Sigma_2}}_{\text{Loop gain}} \leq s, \quad \forall s \in [0, \infty) \quad \Longrightarrow \quad \Sigma \text{ is GAS}$$

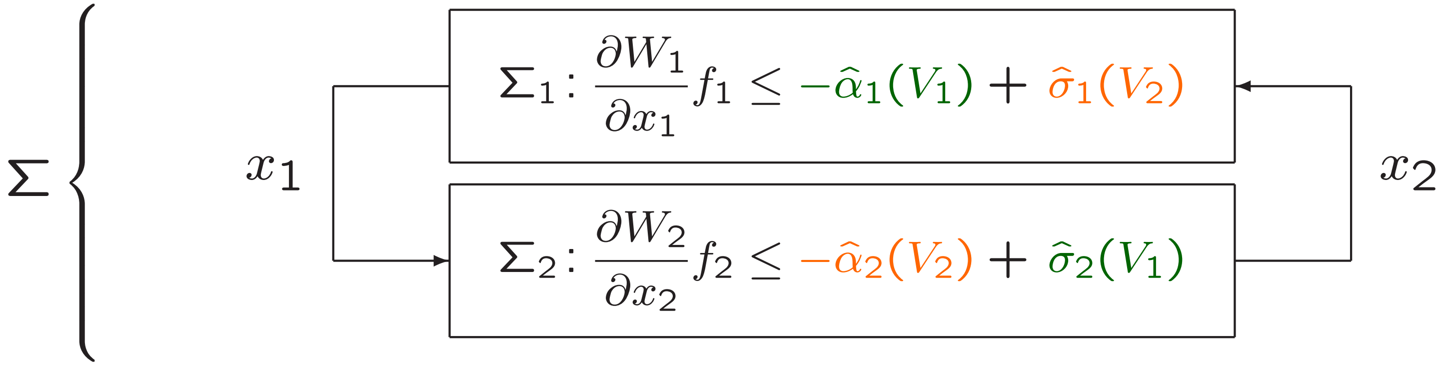
Furthermore, a Lyapunov function is

$$V(x) = \mu_1(V_1(x_1)) + \mu_2(V_2(x_2))$$

where $\mu_i'(s) = \alpha_i(s)^{\varphi} \sigma_{3-i,i}(s)^{\varphi+1}$ with any $\varphi \geq 0$ satisfying

$$\exists \tau \in (1, c] \text{ s.t. } \left(\frac{\tau}{c}\right)^{\varphi+1} < \tau - 1.$$

iISS/ISS Preservation Leads to Small-Gain Theorem



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Remarks on the Small-Gain Theorem

- $\exists c > 1$ s.t. $\gamma_1 \circ \gamma_2(s) \leq s$, $\forall s \in \mathbb{R}_+ \cdots$ (SG)
  is also necessary for the existence of a uniform gain margin.

Remarks on the Small-Gain Theorem

- $\exists c > 1$ s.t. $\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \cdots$ (*SG*)
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Remarks on the Small-Gain Theorem

- $\exists c > 1$ s.t. $\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \cdots$ (*SG*) tight
  is also necessary for the existence of a uniform gain margin.

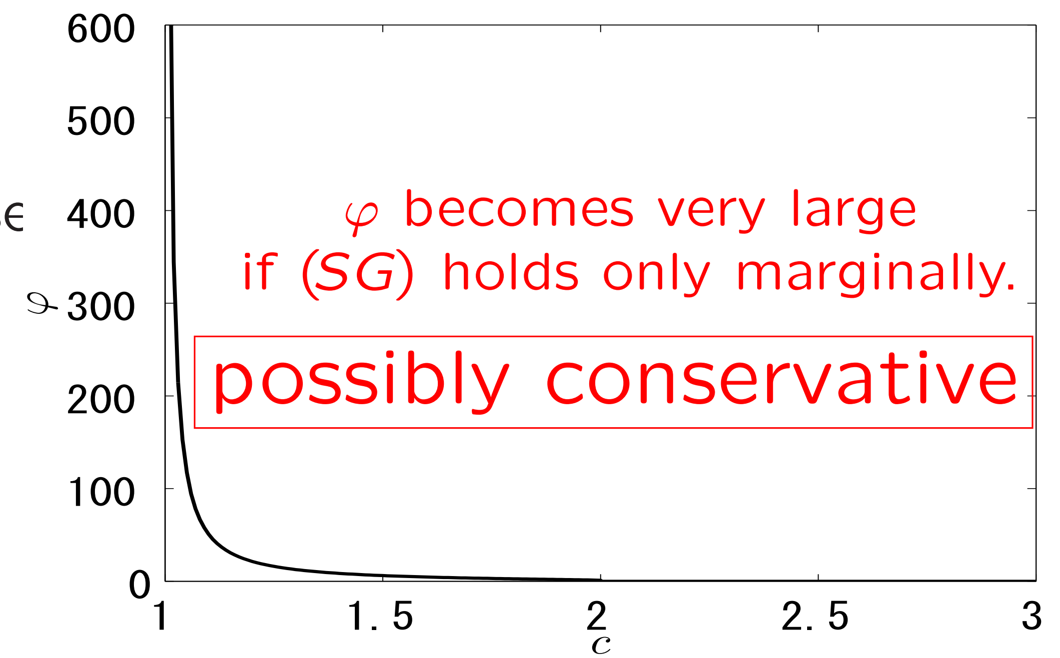
- For the Lyapunov function

$$V(x) = \mu_1(V_1(x_1)) + \mu_2(V_2(x_2)), \quad \mu_i'(s) = \alpha_i(s)^{\varphi}\sigma_{3-i,i}(s)^{\varphi+1},$$

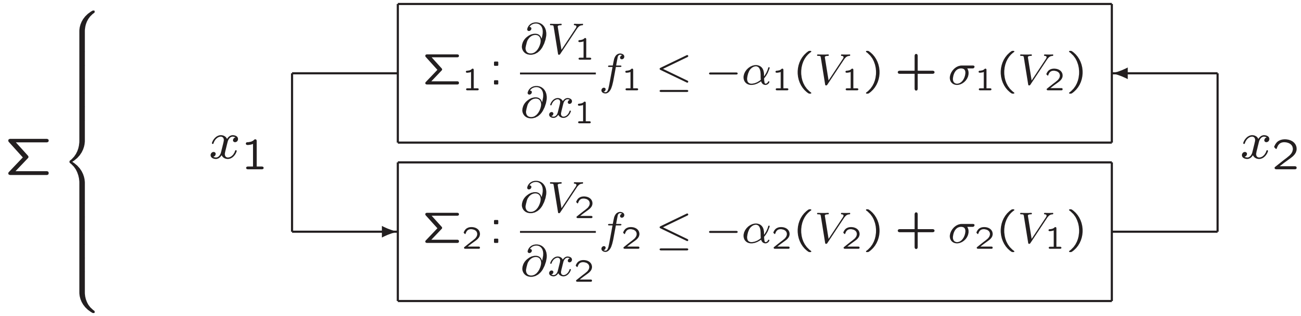
  there always exits $\varphi \geq 0$ satisfying $\exists \tau \in (1, c]$ s.t. $\left(\frac{\tau}{c}\right)^{\varphi+1} < \tau - 1$.

  But, an example is

$$\begin{cases} \varphi = 0 & \text{if } c > 2 \\ \varphi^{-\frac{\varphi}{\varphi+1}} < \frac{c}{\varphi+1} \leq 1 & \text{otherwise} \end{cases}$$



Lyapunov function of Interconnection via LF transform



Theorem Suppose that (SG) holds.

Using $\mu'_{i,\psi} = \alpha_i^{\psi} \sigma_{3-i}^{\psi+1}$ for $i = 1, 2$, define

$\hat{\alpha}_{i,\psi}, \hat{\sigma}_{i,\psi} \cdots\cdots$ LF transform

Then

(i) If $\psi = \varphi$, there exists $\epsilon > 0$ satisfying

$$\hat{\alpha}_{i,\psi}(s) - \hat{\sigma}_{3-i,,\psi}(s) \ge \epsilon\, \mu'_{i,\psi}(s) \alpha_i(s), \ \forall s \in \mathbb{R}_+, i = 1, 2. \qquad \cdots (SP)$$

(ii) If there exist $\psi, \epsilon > 0$ satisfying (SP), then

$$V(x) = \mu_{1,\psi}(V_1(x_1)) + \mu_{2,\psi}(V_2(x_2))$$

establishes GAS of Σ.

Tighter estimates $(\hat{\alpha}_{i,\psi}, \hat{\sigma}_{i,\psi})$ yield smaller ψ.

Benefit of Developed Formulas: Numerical Tractability

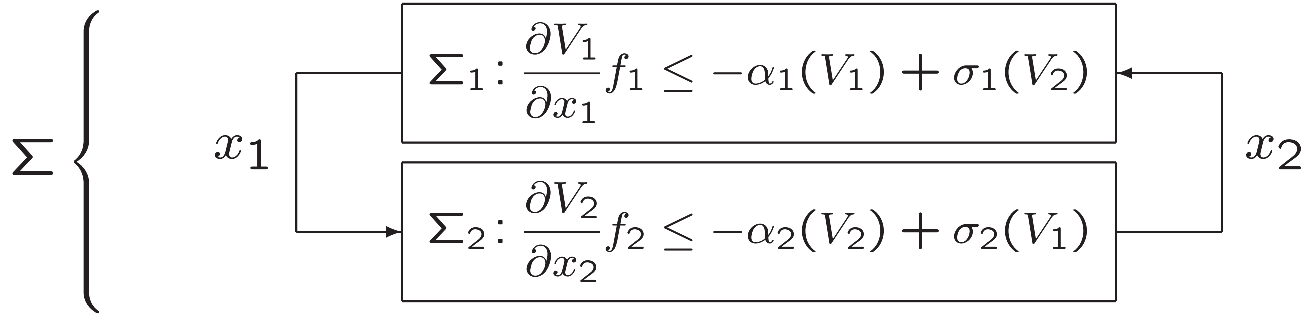
Proposed formulation

- Search for a scalar constant ψ satisfying
$$\hat{\alpha}_{i,\psi}(s) - \hat{\sigma}_{3-i,\psi}(s) \geq \epsilon\, \mu'_{i,\psi}(s)\alpha_i(s),\ \forall s \in \mathbb{R}_+,\ i = 1, 2. \cdots (SP)$$
- The existence of such ψ is guaranteed.
- An upper bound of such ψ is known.
- Increase ψ as $1, 2, 3 \ldots$ until (SP) holds.
  Alternatively, decrease ψ from the upper bound.

Original formulation

- Search for a function $\mu \in \mathcal{S}$ satisfying
$$\sum_{i=1}^{2} \mu'_i(V_i(x_i))\{-\alpha_i(V_i(x_i)) + \sigma_i(V_{3-i}(x_{3-i}))\}$$
$$\leq -\epsilon \sum_{i=1}^{2} \mu'_i(V_i(x_i))\alpha_i(V_i(x_i)),\ \forall (x_1, x_2) \in \mathbb{R}^{N_1+N_2}. \cdots (\mathrm{OP})$$

Example 1: Complexity Reduction of V by LF Transform



iISS-ISS interconnection: $\alpha_1(s)=\frac{s}{s+1}$, $\sigma_1(s)=\frac{5s}{4s+6}$, $\alpha_2(s)=s$, $\sigma_2(s)=\frac{5s}{7s+6}$

$$V(x)=\int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{26} \left[\frac{5s}{7s+6}\right]^{27} ds + \int_0^{V_2(x_2)} s^{26} \left[\frac{5s}{4s+6}\right]^{27} ds$$

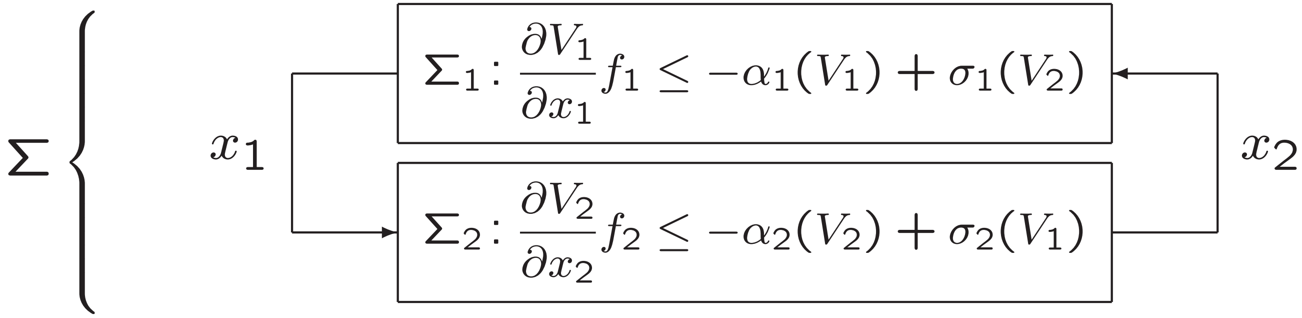
$$\downarrow$$

$$V(x)=\int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{2} \left[\frac{5s}{7s+6}\right]^{3} ds + \int_0^{V_2(x_2)} s^{2} \left[\frac{5s}{4s+6}\right]^{3} ds$$

Σ_1: LF transform

Σ_2: LF transform

Example 1: Complexity Reduction of V by LF Transform



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$\downarrow$

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Σ_1: LF transform
Σ_2: LF transform

cf. $$V(x)=\int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{5} \left[\frac{5s}{7s+6}\right]^{6} ds + \int_0^{V_2(x_2)} s^{5} \left[\frac{5s}{4s+6}\right]^{6} ds$$

Σ_1: –
Σ_2: LF transform

Example 2: Complexity Reduction of V by LF Transform

$$\Sigma \begin{cases} \Sigma_1 : \dfrac{\partial V_1}{\partial x_1} f_1 \le -\alpha_1(V_1) + \sigma_1(V_2) \\ \Sigma_2 : \dfrac{\partial V_2}{\partial x_2} f_2 \le -\alpha_2(V_2) + \sigma_2(V_1) \end{cases}$$

x_1 x_2

ISS-ISS interconnection: $\alpha_1(s) = \dfrac{s}{s+1}$, $\sigma_1(s) = \dfrac{4s}{5(s+1)}$, $\alpha_2(s) = s$, $\sigma_2(s) = \dfrac{8s}{5(s+2)}$

$$V(x) = \int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{39} \left[\frac{8s}{5(s+2)}\right]^{40} ds + \int_0^{V_2(x_2)} s^{39} \left[\frac{4s}{5(s+1)}\right]^{40} ds$$

$$\downarrow$$

$$V(x) = \int_0^{V_1(x_1)} \left[\frac{s}{s+1}\right]^{1} \left[\frac{8s}{5(s+2)}\right]^{2} ds + \int_0^{V_2(x_2)} s^{1} \left[\frac{4s}{5(s+1)}\right]^{2} ds$$

Example 3: ISS Preservation is Not Necessary for Interconnection

ISS-ISS interconnection: $\underbrace{\alpha_1(s)=\frac{s}{s+1},\ \sigma_2(s)=\frac{4s}{5s+5}}_{\text{marginally ISS}}^{\Sigma_1},\ \underbrace{\alpha_2(s)=s,\ \sigma_1(s)=\frac{s}{4}}_{\text{ISS by a large margin}}^{\Sigma_2}$

Scalings ························ μ_1 preserving ISS μ_2 preserving ISS

$$V(x)=\int_0^{V_1(x_1)}\left[\frac{s}{s+1}\right]^{17}\left[\frac{s}{4}\right]^{18}ds+\int_0^{V_2(x_2)}s^{17}\left[\frac{4s}{5s+5}\right]^{18}ds$$

Scalings ························ μ_1 preserving iISS μ_2 preserving ISS

$$V(x)=\int_0^{V_1(x_1)}\left[\frac{s}{s+1}\right]^{0}\left[\frac{s}{4}\right]^{1}ds+\int_0^{V_2(x_2)}s^{0}\left[\frac{4s}{5s+5}\right]^{1}ds$$

Key idea of iISS small-gain theorem beyond ISS (Ito&Jiang'09).

- Only one of the two systems Σ_i is necessarily ISS.
- One system can compensate weak stability of the other system.

ISS Preservation is Not Necessary for Interconnection (cont'd)

Key idea of iISS small-gain theorem beyond ISS (Ito&Jiang'09).

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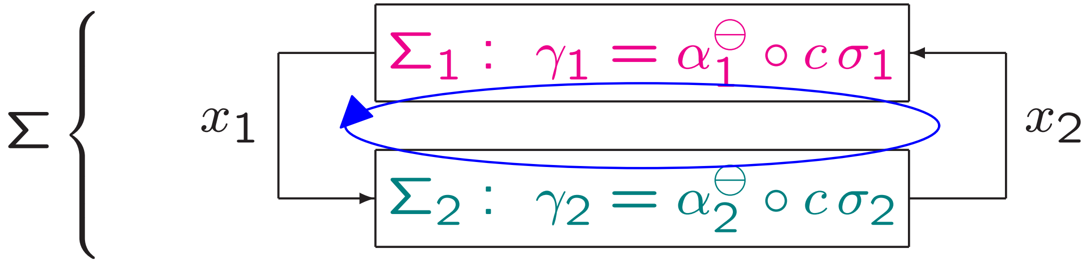
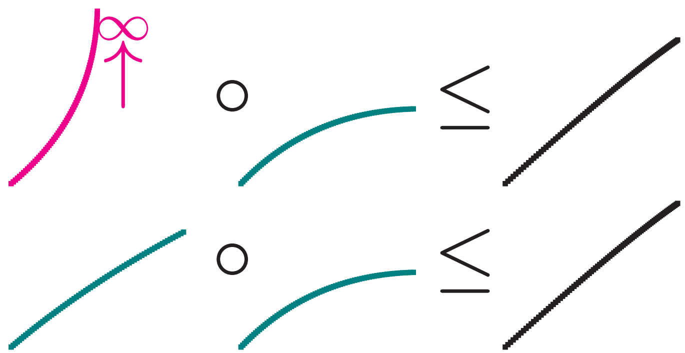


There exists $c > 1$ such that

$$\underbrace{\gamma_1 \circ \gamma_2(s)}_{\text{Loop gain}} \le s, \quad \forall s \in [0, \infty) \quad \Longrightarrow \quad \Sigma \text{ is GAS}$$



Conclusions

Manipulating dissipation inequalities by scaling Lyapunov functions

- Flexibility reduces greatly when decay rate is radially vanishing.
- Explicit formulas to preserve iISS/ISS dissipation inequalities.

  ··· Division approach

  ··· LF transformation approach
- The division approach is a special case of the LF transform.
- The iISS small-gain theorem is revisited via the LF transform.
  - Complexity of Lyapunov functions can be greatly reduced.
  - The reduction is beneficial in controller design.
  - ISS preservation of all components is not necessary.

[1] H.Ito and C.M.Kellett, “Preservation and interconnection of iISS and ISS dissipation inequalities by scaling”, The 1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems (MICNON), pp.776-781, Saint Petersburg, Russia, June 25, 2015.

[2] H.Ito and C.M.Kellett, “iISS and ISS Dissipation Inequalities: Preservation and Interconnection by Scaling”, submitted in July 7, 2015.