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Stabilization of rigid formation and open problems

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Multi-agent systems & Distributed formation control

Agents and multi-agent systems:

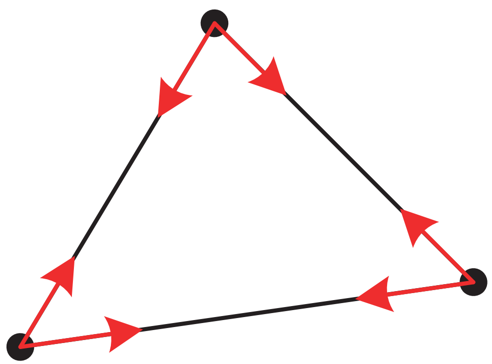
- An agent is understood as a dynamical system.
- A multi-agent system is a collection, a group, or a team of dynamical systems.

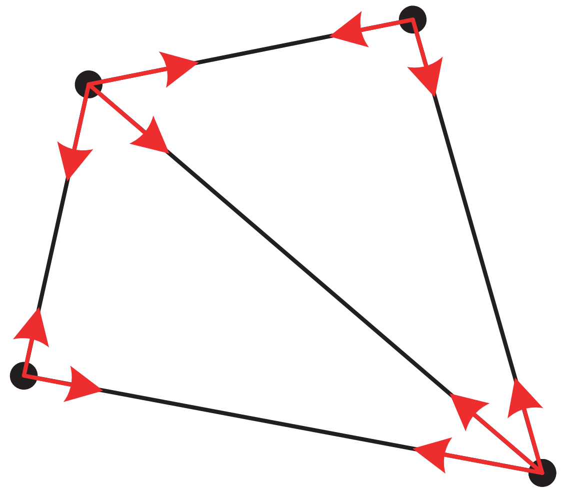
Distributed formation control:

- No centralized controller for a given multi-agent system.
- Each agent has its own controller based on interaction with its neighboring agents.
- Only the distances among agents are controlled by relative interactions;
  $\rightarrow$ but a formation defined w.r.t a global coordinate frame is achieved.

Problem statement

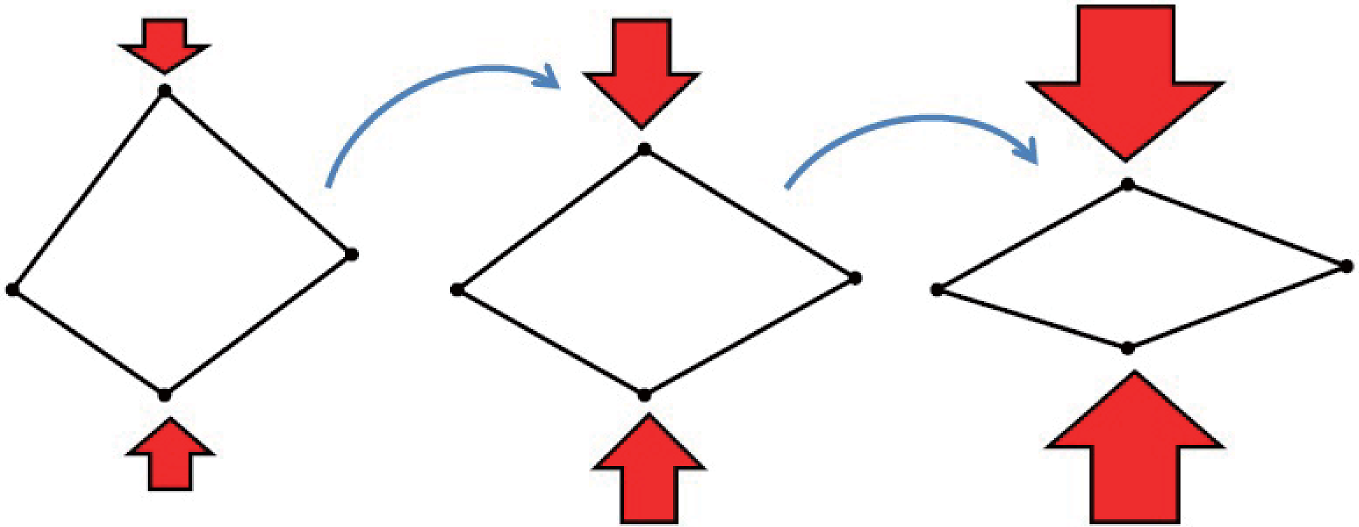
- Only local relative measurements
- Each node controls its neighbor edges only
- Control strategy for individual nodes?
- What are properties of graph for unique formation?





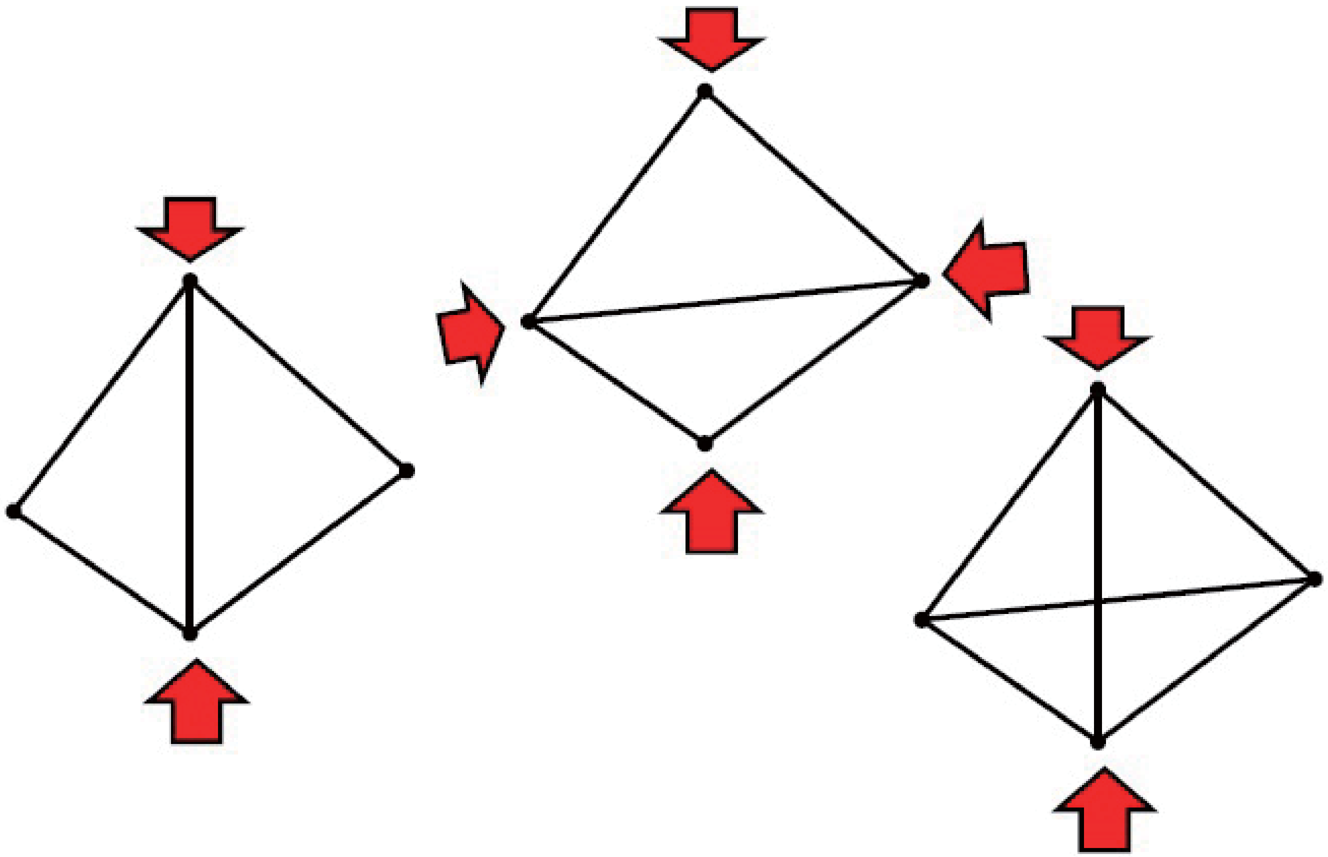
Problem statement

- Not rigid (flex)
- Distances are fixed; but configuration is changed with external forces



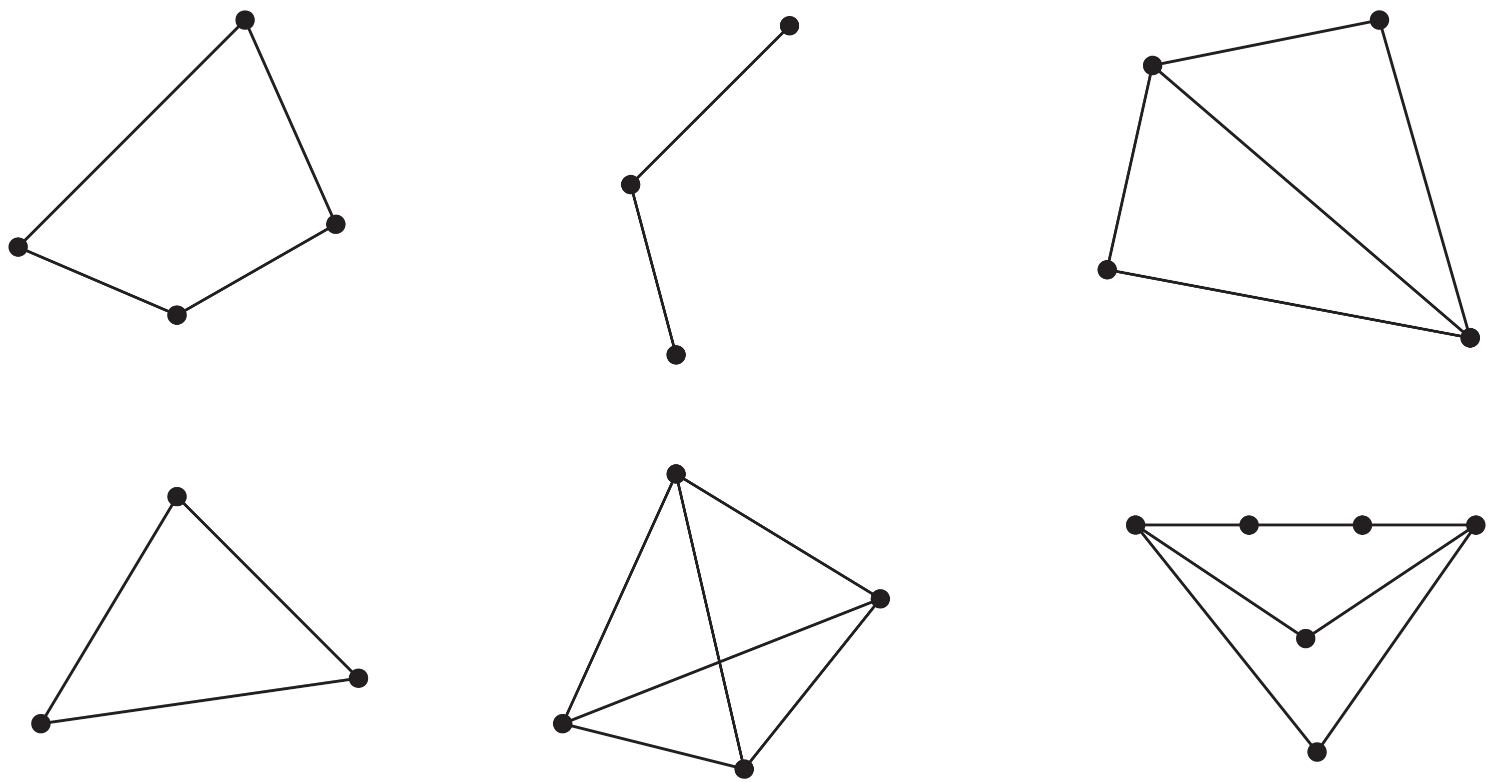
Problem statement

- Rigid
- Configuration does not change provided that the distances are fixed even with external forces



Problem statement

- Only distances are constrained
- Formation is fixed (rigid) or not-fixed (flex) ?



Problem statement

- Agent model:

$$\dot{p}_i = u_i, \; i = 1, \ldots, N,$$

where $p_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$.
- Interaction graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Sensed variables:

$$p_{ji}^i = p_j^i - p_i^i, \; j \in \mathcal{N}_i, \; i \in \mathcal{V},$$

where the superscript i denotes that the variables are with respect to the local reference frames of agent i, and $\mathcal{N}_i$ is the set of all neighbors of agent i.

- Overall task: Given $p^* = (p_1^{*T}, \ldots, p_N^{*T})^T$,

$$\forall i, j \in \mathcal{V}, \; \|p_i - p_j\| \to \|p_i^* - p_j^*\|.$$

- Local task for agent i:

$$\forall j \in \mathcal{N}_i, \; \|p_i - p_j\| \to \|p_i^* - p_j^*\|.$$

- Desired invariant set:

$$E_{p^*} \triangleq \{p : \|p_i - p_j\| = \|p_i^* - p_j^*\|\}.$$

- Also, ensure $\dot{p}_i \to 0$ or $\dot{p}_i < \infty$.

Graph rigidity

Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$, let us assign $p_i \in \mathbb{R}^n$ to each vertex i for all $i \in \mathcal{V}$.

- Realization: $p = (p_1^T, \ldots, p_N^T)^T \in \mathbb{R}^{nN}$, Framework: $(\mathcal{G}, p)$
- Equivalence: Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are equivalent if

$$\forall\ (i,j) \in \mathcal{E},\ \ \|p_i - p_j\| = \|q_i - q_j\|.$$

- Congruence: Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are congruent if

$$\forall\ i,j \in \mathcal{V},\ \ \|p_i - p_j\| = \|q_i - q_j\|.$$

Definition (Rigidity)

A framework $(\mathcal{G}, p)$ is rigid if there exists a neighborhood U_p of p such that all frameworks equivalent to $(\mathcal{G}, p)$ are congruent in U_p.

☞ If $(\mathcal{G}, p)$ is rigid, then the overall task and the local tasks are consistent.

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Preliminaries: incident matrices

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

- Incidence matrix: $H = [h_{ij}] \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{V}|}$

$$h_{ij} \triangleq \begin{cases} 1, & \text{if vertex } j \text{ is the sink vertex of edge } i, \\ -1, & \text{if vertex } j \text{ is the source vertex of edge } i, \\ 0, & \text{otherwise;} \end{cases}$$

- Edge partitioning: $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$, where $\mathcal{E}_+$ and $\mathcal{E}_-$ are disjoint and $(i,j) \in \mathcal{E}_+$ implies $(j,i) \in \mathcal{E}_-$.
- Incidence matrix partitioning: $H = [H_+^T, -H_+^T]^T$, where H_+ is the incidence matrix corresponding to $\mathcal{E}_+$.
- Link: the link $e = (e_1, \ldots, e_{M/2}) \in \mathbb{R}^{n(M/2)}$, $e_i \in \mathcal{E}_+$, of a framework $(\mathcal{G}, p)$ is defined as ($e_k = p_i - p_j; k = (i,j)$):

$$e \triangleq (H_+^T \otimes I_n)p.$$

Link space

- Notations $\hat{H}_+ = H_+ \otimes I_n$. In undirected graph (under gradient control setups), we use $\hat{H} = \hat{H}_+ = H_+ \otimes I_n = \hat{H}_- = H_- \otimes I_n$, and $M/2 = m$ (i.e., cardinality of edges in undirected graph).
- Link space: The space $\mathrm{Im}(H_+^T \otimes I_n)$ is referred to as the link space associated with the framework $(\mathcal{G}, p)$.
- Edge function: We define a function $v_{\mathcal{G}} : \mathrm{Im}(H_+^T \otimes I_n) \to \mathbb{R}^{M/2}$ as

$$v_{\mathcal{G}}(e) \triangleq (\|e_1\|^2, \ldots, \|e_{M/2}\|^2),$$

  which corresponds to the edge function $g_{\mathcal{G}}$ parameterized in the link space. That is, $g_{\mathcal{G}}(p) = v_{\mathcal{G}}((H_+^T \otimes I_n)p)$.
- Defining D as $D(e) \triangleq \mathrm{diag}(e_1, \ldots, e_{M/2})$, we obtain

$$\frac{\partial g_{\mathcal{G}}(p)}{\partial p} = \frac{\partial v_{\mathcal{G}}(e)}{\partial e}\frac{\partial e}{\partial p} = [D(e)]^T (H_+^T \otimes I_n).$$

Gradient control laws - Krick, Broucke & Francis, 2009

- A potential function $\phi(p)$ as a function of $g_{\mathcal{G}} - d^*$

$$\phi(p) = \frac{1}{2}\|g_{\mathcal{G}} - d^*\|$$

- With $u = -(\nabla\phi(p))^T$,

$$\dot{p} = -H_+^T J_v^T (v_{\mathcal{G}}(e) - d^*)$$

where $J_v = 2\text{diag}\{e_i^T\}$.

- Control law for each agent is

$$\dot{p}_i = u_i = -\sum_{j \in \text{edges leaving } i} \frac{1}{2}(\|e_j\|^2 - d_j^*)e_j$$

Gradient control laws - Krick, Broucke & Francis, 2009

- The centroid $p^o = \frac{1}{n}\sum_{i=1}^{n} p_p$ is stationary: i.e., $\dot{p}^o = 0$.
- Conduct coordinate transformation

$$\tilde{p} = \begin{bmatrix} p^o \\ \bar{p} \end{bmatrix} = \mathbf{P}p \tag{1}$$

where $\mathbf{P}$ is an orthonormal matrix whose first two rows are $\frac{1}{n}\mathbf{1}^T \otimes I_2$.

- Equilibria

$$\begin{aligned} \mathcal{E}_1 &:= \{p | g(p) - d^* = 0\} = \{p | \phi(p) = 0\} \\ \mathcal{E}_2 &:= \{p | J_v^T (g(p) - d^*) = 0\} \\ \mathcal{E} &:= \{p | \nabla\phi(p) = 0\} \end{aligned}$$

It is noticeable that $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}$. The matrix H_+^T is $2n \times 2m$, so if $m > n$, the it has a nontrivial kernel.

Gradient control laws - Krick, Broucke & Francis, 2009

- It is also possible to define equilibrium sets (target formations) for the reduced state $\bar{p}$ such as

$$\bar{\mathcal{E}}_1 := \{p \in \mathbb{R}^{2N-2} | v(\bar{H}\bar{p}) - d^* = 0\}$$

- The advantage of using $\bar{\mathcal{E}}_1$ rather than $\mathcal{E}_1$ in the ensuing stability analysis is that $\bar{\mathcal{E}}_1$ is compact, whereas $\mathcal{E}_1$ is not.
- Key idea: Via linearization $\Longrightarrow$ Center manifold theory

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Motivation & objective

Assumptions:

- $(\mathcal{G}, p^*)$ is infinitesimally rigid.
- Realization dimension: general n-dimension.
- Control law: generalized version of the gradient control law [Baillieul & Suri, 2003].

Objectives:

- Lyapunov stability analysis of rigid formations of single-integrators in n-dimensional space.
- Extension of the result to double-integrator formations.

Generalized gradient control law

- Global potential function ϕ:

$$\phi(p) \triangleq \frac{k_p}{2} \sum_{(i,j)\in\mathcal{E}_+} \gamma\left(\|p_j - p_i\|^2 - d_{ji}^*\right),$$

where $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is positive definite and analytic in some neighborhood of 0.

- Gradient control law:

$$\dot{p} = u = -\nabla\phi(p) = -k_p\left(H_+ \otimes I_n\right) D(e)\Gamma(\tilde{d}), \quad (2)$$

where $e \triangleq (H_+^T \otimes I_n)p$, $\tilde{d} = (\|e_1\|^2 - \|e_1^*\|^2, \ldots, \|e_{M/2}\|^2 - \|e_{M/2}^*\|^2)$ and $\Gamma(\tilde{d}) \triangleq \left(\frac{\partial\gamma(\tilde{d}_1)}{\partial\tilde{d}_1}, \ldots, \frac{\partial\gamma(\tilde{d}_{M/2})}{\partial\tilde{d}_{M/2}}\right)$.

Generalized gradient control law

- The gradient system is now described in the link space as follows:

$$\begin{aligned}\dot{e} &= (H_+^T \otimes I_n)\dot{p} \\ &= -k_p (H_+^T \otimes I_n)(H_+ \otimes I_n) D(e) \Gamma(\tilde{d})\end{aligned}$$

  - For a given realization $p^* = [p_1^{*T} \cdots p_N^{*T}]^T \in \mathbb{R}^{nN}$, we define the desired formation E_{p^*} of the agents as the set of formations that are congruent to p^*:

$$E_{p^*} := \{p \in \mathbb{R}^{nN} : \|p_j - p_i\| = \|p_j^* - p_i^*\|, \ \forall i,j \in \mathcal{V}\}. \qquad (3)$$

  - Equilibrium set in position

$$E'_{p^*} = \{p \in \mathbb{R}^{nN} : \|p_j - p_i\| = \|p_j^* - p_i^*\|, \forall (i,j) \in \mathcal{E}_+\}$$

  - Equilibrium set in the link space (compact)

$$E'_{e^*} = \{e \in \mathrm{Im}(H_+^T \otimes I_n) : \|e_i\| = \|e_i^*\|, \forall i = 1, \ldots, m\}$$

Generalized gradient control law

- Main idea: $E'_{e^*} \Rightarrow E'_{p^*} \Rightarrow E_{p^*}$ or $E'_{e^*} \Rightarrow E'_{p^*} \Leftrightarrow E_{p^*}$ or $E'_{e^*} \Leftrightarrow E'_{p^*} \Leftrightarrow E_{p^*}$
- To analyze the stability of E'_{e^*} , we define $V : \mathrm{Im}(H_+^T \otimes I_n) \rightarrow \bar{\mathbb{R}}_+$ as

$$V(e) := \sum_{i=1}^{M} \frac{1}{2}\gamma\left(\|e_i\|^2 - \|e_i^*\|^2\right).$$

- The time-derivative of V can be arranged as

$$\begin{aligned}
\dot{V}(e) &= \frac{\partial V(e)}{\partial e}\dot{e} = -k_p \frac{\partial V(e)}{\partial e}(H_+^T \otimes I_n)(H_+ \otimes I_n) D(e)\Gamma(\tilde{d}) \\
&= -k_p \underbrace{\left[D(e)\Gamma(\tilde{d})\right]^T (H_+ \otimes I_n)^T}_{=-[\nabla\phi(p)]^T} \underbrace{(H_+ \otimes I_n) D(e)\Gamma(\tilde{d})}_{=-\nabla\phi(p)} \\
&= -k_p \|\nabla\phi(p)\|^2 \leq 0,
\end{aligned}$$

which shows the local stability of E'_{e^*}.

Generalized gradient control law

- Then the local asymptotic stability of E'_{e*} can be ensured by showing the existence of a neighborhood $U_{E'_{e*}}$ of E'_{e*} such that, for any $e \in U_{E'_{e*}}$, if $e \notin E_{e^*}$ (or, $e \notin E'_{e^*}$), then $\dot{V}(e) < 0$.

Theorem

(Lojasiewicz's inequality) Suppose that $f : D \subseteq \mathbb{R}^{n_f} \to \mathbb{R}$ is a real analytic function in a neighborhood of $z \in D$. There exist constants $k_f > 0$ and $\rho_f \in [0, 1)$ such that

$$\|\nabla f(x)\| \geq k_f \|f(x) - f(z)\|^{\rho_f}$$

in some neighborhood of z.

Generalized gradient control law

Lemma

For any $\bar{p} \in E'_{p^*}$, *there exists a neighborhood* $U_{\bar{p}}$ *of* $\bar{p}$ *such that, for any* $p \in U_{\bar{p}}$ *and* $p \notin E'_{p^*}$, $\|\nabla\phi(p)\| > 0$.

Proof.

Since γ is analytic in some neighborhood of 0, for any $\bar{p} \in E'_{p^*}$, there exists a neighborhood of $\bar{p}$ such that ϕ is analytic in the neighborhood. Thus it follows from *Theorem 2* that there exist $k_\phi > 0$, $\rho_\phi \in [0, 1)$, and a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that

$$\|\nabla\phi(p)\| \geq k_\phi \|\phi(p) - \phi(\bar{p})\|^{\rho_\phi} = k_\phi \|\phi(p)\|^{\rho_\phi}.$$

for all $p \in U_{\bar{p}}$. Further, $\phi(p) = 0$ if and only if $p \in E'_{p^*}$ by the positive definiteness of γ. Thus, for any $p \in U_{\bar{p}}$ and $p \notin E'_{p^*}$, $\|\nabla\phi(p)\| > 0$. □

Generalized gradient control law

Lemma

For any $\bar{p} \in E'_{p^*}$*, there exists a neighborhood* $U_{\bar{p}}$ *of* $\bar{p}$ *such that, for any* $p \in U_{\bar{p}}$ *and* $p \notin E'_{p^*}$*,* $\|\nabla\phi(p)\| > 0$*.*

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Generalized gradient control law

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Generalized gradient control law

The local asymptotic stability of E'_{p*} is then ensured based on *Lemma 6* as follows:

Theorem

The set E'_{p*} is locally asymptotically stable with respect to (2).

Proof.

We prove this theorem by showing that E'_{e^*} is locally asymptotically stable.
To show the local asymptotic stability of E'_{e^*}, we construct a neighborhood of E'_{e^*} such that $\dot{V}(e) \geq 0$ for any e in the neighborhood and $\dot{V}(e) = 0$ if and only if $e \in E'_{e^*}$.
It follows from *Lemma 6* that, for any $\bar{p} \in E'_{p^*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that $\|\nabla\phi(p)\| > 0$ for all $p \in U_{\bar{p}}$ and $p \notin E'_{p^*}$. We take $r_p^* > 0$ such that

$$D_{r_p^*} := \{p \in \mathbb{R}^{nN} : \|p - \bar{p}\| < r_p^*\} \subseteq U_{\bar{p}}.$$

Define

$$U_{E'_{e^*}}(r_e) := \{e \in \mathrm{Im}(H_+^T \otimes I_n) : \inf_{\eta \in E'_{e^*}} \|e - \eta\| < r_e\}.$$

Let $r_e^* = \sigma_{min}(H_+^T \otimes I_n) r_p^*$, where $\sigma_{min}(H_+^T \otimes I_n)$ denotes the non-zero smallest singular value of $H_+^T \otimes I_n$.

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Proof.

(Cont.) Then, for any $e \in U_{E'_{e*}}(r_e^*)$, there exists $\bar{e} \in E'_{e*}$ such that

$$\inf_{\eta \in E'_{e*}} \|e - \eta\| = \|e - \bar{e}\| < r_e^*$$

because E'_{e*} is compact and $\|e - \eta\|$ is a continuous function of η. From the fact that $(e - \bar{e}) \in \mathrm{Im}(H_+^T \otimes I_n)$, there always exist $p \in \mathbb{R}^{nN}$ and $\bar{p} \in E'_{p*}$ such that $(H_+^T \otimes I_n)(p - \bar{p}) = e - \bar{e}$ and $(p - \bar{p}) \in \mathrm{Im}(H_+^T \otimes I_n)$. Since $p - \bar{p}$ belongs to the row space of $H_+^T \otimes I_n$, we obtain

$$\sigma_{min}(H_+^T \otimes I_n)\|p - \bar{p}\| \leq \|e - \bar{e}\|$$

Thus we have $\|p - \bar{p}\| \leq \frac{\|e - \bar{e}\|}{\sigma_{min}(H_+^T \otimes I_n)} < r_p^*$, which implies that $p \in U_{\bar{p}}$ from (4). It follows from *Lemma 6* that if $e \notin E'_{e*}$, $\dot{V}(e) = -k_p\|\nabla\phi(p)\|^2 < 0$, which implies that E'_{e*} is locally asymptotically stable. Thus E'_{p*} is locally asymptotically stable with respect to (2). □

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Thus we have $\|p - \bar{p}\| \leq \frac{\|e - \bar{e}\|}{\sigma_{min}(H_+^T \otimes I_n)} < r_p^*$, which implies that $p \in U_{\bar{p}}$ from (4). It follows from *Lemma 6* that if $e \notin E'_{e*}$, $\dot{V}(e) = -k_p \|\nabla \phi(p)\|^2 < 0$, which implies that E'_{e*} is locally asymptotically stable. Thus E'_{p*} is locally asymptotically stable with respect to (2). □

Proof.

(Cont.) Then, for any $e \in U_{E'_{e*}}(r_e^*)$, there exists $\bar{e} \in E'_{e*}$ such that

$$\inf_{\eta \in E'_{e*}} \|e - \eta\| = \|e - \bar{e}\| < r_e^*$$

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Proof.

(Cont.) Then, for any $e \in U_{E'_{e*}}(r^*_e)$, there exists $\bar{e} \in E'_{e*}$ such that

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because E'_{e*} is compact and $\|e - \eta\|$ is a continuous function of η. From the fact that $(e - \bar{e}) \in \mathrm{Im}(H^T_+ \otimes I_n)$, there always exist $p \in \mathbb{R}^{nN}$ and $\bar{p} \in E'_{p*}$ such that $(H^T_+ \otimes I_n)(p - \bar{p}) = e - \bar{e}$ and $(p - \bar{p}) \in \mathrm{Im}(H^T_+ \otimes I_n)$. Since $p - \bar{p}$ belongs to the row space of $H^T_+ \otimes I_n$, we obtain

$$\sigma_{min}(H^T_+ \otimes I_n)\|p - \bar{p}\| \leq \|e - \bar{e}\|$$

Thus we have $\|p - \bar{p}\| \leq \frac{\|e - \bar{e}\|}{\sigma_{min}(H^T_+ \otimes I_n)} < r^*_p$, which implies that $p \in U_{\bar{p}}$ from (4). It follows from *Lemma 6* that if $e \notin E'_{e*}$, $\dot{V}(e) = -k_p\|\nabla\phi(p)\|^2 < 0$, which implies that E'_{e*} is locally asymptotically stable. Thus E'_{p*} is locally asymptotically stable with respect to (2). □

Stability analysis

Theorem

If $(\mathcal{G}, p^*)$ is rigid, the set E_{p^*} is locally asymptotically stable with respect to (2).

Proof.

From *Theorem 7*, E'_{p^*} is locally asymptotically stable. Since $(\mathcal{G}, p^*)$ is rigid, it follows from the definition of the graph rigidity that, for any $\bar{p} \in E_{p^*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that $E_{p^*} \cap U_{\bar{p}} = E'_{p^*} \cap U_{\bar{p}}$. This implies that E_{p^*} is locally asymptotically stable with respect to (2). □

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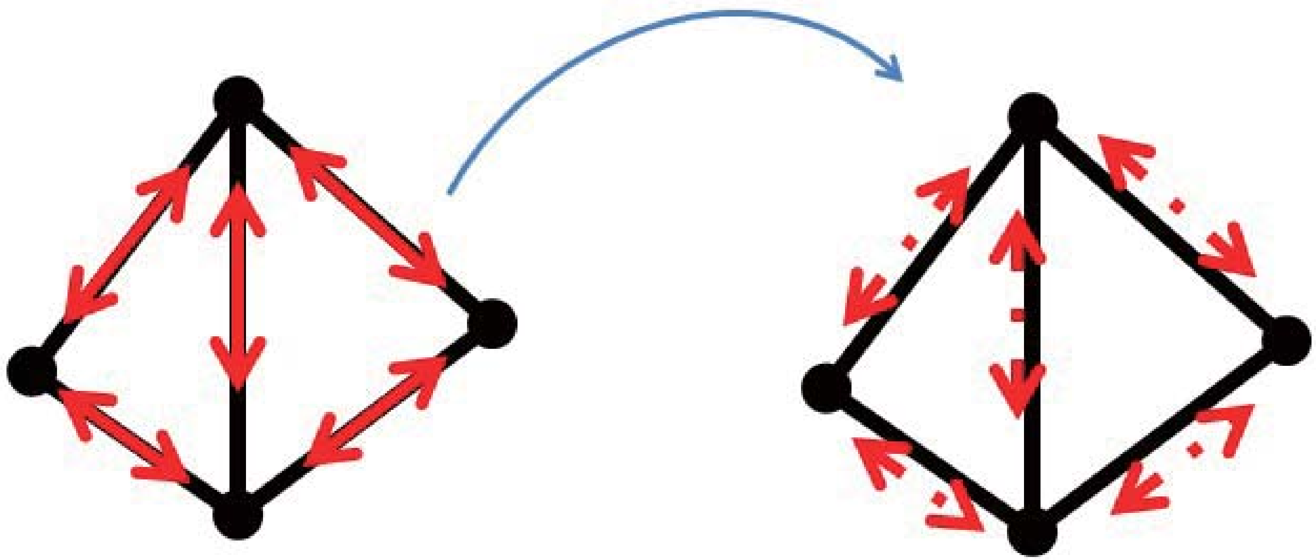
Main idea

- Edges (inter-agent distances) are analyzed as control inputs
- Then, the control inputs for edges are separated into neighbor agents



Inter-agent distance dynamics

The time-derivative of $d_{ij}(\triangleq \|p_i - p_j\|^2)$ for any $(i,j) \in \mathcal{E}$:

$$\dot{d}_{ij} = \frac{d}{dt}(\|p_i - p_j\|^2) = \underbrace{2(p_i - p_j)^T(u_i - u_j)}_{\text{Virtual control law } u_{ij} \triangleq}.$$

Design procedure

(1) Design u_{ij} to stabilize d_{ij} such that $d_{ij} \to d_{ij}^*$; (2) Then design u_i and u_j to implement u_{ij}.

- Virtual control input design:

$$u_{ij} = -k_d(d_{ij} - d_{ij}^*) \Rrightarrow d_{ij}(t) = e^{-k_d t} d_{ij}^0 + (1 - e^{-k_d t}) d_{ij}^*.$$

- Virtual control law vs. control law for the agents,

$$u_{ij} = \underbrace{2(p_i - p_j)^T(u_i - u_j)}_{\text{By definition}} = \underbrace{-k_d \tilde{d}_{ij}}_{\text{By design}}, \; \tilde{d}_{ij} = d_{ij} - d_{ij}^*.$$

Three-agent case

Proposed control law for three-agent case:

$$\begin{array}{r} 4(p_j - p_i)^T u_i = k_d \tilde{d}_{ij}, \\ 4(p_i - p_j)^T u_j = k_d \tilde{d}_{ij} \end{array} \Rightarrow \underbrace{\begin{pmatrix} (p_j - p_i)^T \\ (p_k - p_i)^T \end{pmatrix}}_{A_i \triangleq} u_i = \frac{k_d}{4} \underbrace{\begin{pmatrix} \tilde{d}_{ij} \\ \tilde{d}_{ik} \end{pmatrix}}_{b_i \triangleq} \Rightarrow u_i = \frac{k_d}{4} A_i^{-1} b_i.$$

Theorem

For three-agents in the plane, if p^0 and p^* are not collinear, then

- *the proposed control law is nonsingular;*
- *the invariant set E_p^* is globally asymptotically stable;*
- *$\tilde{d}_{ij}$ for all $(i,j) \in \mathcal{E}$ exponentially and monotonically converge to zero.*

General case

Given $d = (\ldots, d_{ij}, \ldots)$ for all $(i,j) \in \mathcal{E}$, $(\mathcal{G}, d)$ is realizable if there exists a realization $(p_1^T, \ldots, p_N^T)^T \in \mathbb{R}^{nN}$ such that $\forall i,j \in \mathcal{V}$, $\|p_i - p_j\|^2 = d_{ij}$.

Realizability problem

$(\mathcal{G}, d^0)$ and $(\mathcal{G}, d^*)$ is realizable in two-dimension $\Rightarrow$ $(\mathcal{G}, \alpha d^0 + (1-\alpha)d^*)$, where $0 \leq \alpha \leq 1$, is realizable in at most four-dimension [Havel *et al.*, 1983].

☞ No control law such that $u_{ij} = -k_d(d_{ij} - d_{ij}^*)$.

The virtual control law $u_{ij} = -k_d(d_{ij} - d_{ij}^*)$ gives rise to a possibly over-determined system of linear equations

$$\underbrace{\begin{pmatrix} \vdots \\ (p_j - p_i)^T \\ \vdots \end{pmatrix}}_{A_i \triangleq} u_i = \frac{k_d}{4} \underbrace{\begin{pmatrix} \vdots \\ \tilde{d}_{ij} \\ \vdots \end{pmatrix}}_{b_i \triangleq}, \; j \in \mathcal{N}_i,$$

Projection of $\frac{k_d}{4} b_i$ to the column space of A_i
$\Leftrightarrow$ Projection of the realization of $(\mathcal{G}, \alpha d^0 + (1-\alpha)d^*)$ to the plane

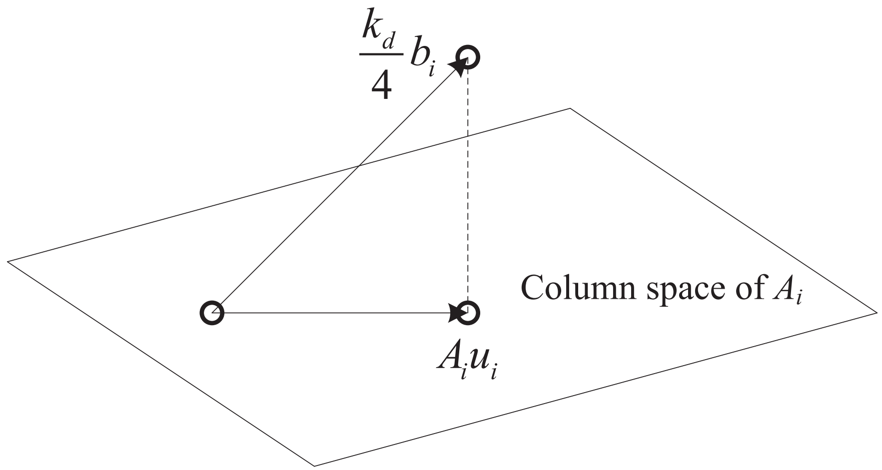
General case

Proposed control law:

$$A_i u_i = \frac{k_d}{4} b_i$$

$$\Rightarrow u_i = \underset{u_i \in \mathbb{R}^2}{\operatorname{argmin}} \|A_i u_i - \frac{k_d}{4} b_i\|^2$$

$$\Rightarrow u_i = \frac{k_d}{4} (A_i^T A_i)^{-1} A_i^T b_i \tag{5}$$



Lemma

(Used for ensuring existence of control input) For N-agents, if $(\mathcal{G}, p)$ is infinitesimally rigid in the plane, then the proposed control law is nonsingular.

Proof.

Due to the infinitesimal rigidity of $(\mathcal{G}, p)$, the first leading principal minor of $A_i^\top A_i$ is positive: $\sum_{j\in\mathcal{N}_i}(x_j - x_i)^2 > 0$ for all $i \in \mathcal{V}$. Since $N \geq 3$ and agent i has at least two neighboring agents due to the rigidity of $(\mathcal{G}, p)$, the second leading principal minor of $A_i^\top A_i$ is also positive by the Cauchy-Schwarz inequality:

$$\sum_{j\in\mathcal{N}_i}(x_j - x_i)^2 \sum_{j\in\mathcal{N}_i}(y_j - y_i)^2 - \left(\sum_{j\in\mathcal{N}_i}(x_j - x_i)(y_j - y_i)\right)^2 > 0.$$

The second leading principal minor of $A_i^\top A_i$ is zero if and only if $(\ldots, x_j - x_i, \ldots)$ and $(\ldots, y_j - y_i, \ldots)$, $j \in \mathcal{N}_i$, are linearly dependent, which implies that p_i and p_j, $j \in \mathcal{N}_i$, are collinear. It then follows from Sylvester's criterion that $A_i^\top A_i$ is positive definite. Thus $(A_i^\top A_i)^{-1}$ is positive definite by the positive definiteness of $A_i^\top A_i$.

□

General case

Lemma

(Used for proving negative definiteness of the derivative of Lyapunov function) Given an N-agent group, if $(\mathcal{G}, p^*)$ is infinitesimally rigid, then there exists a level set $\Omega_c = \{e : V(e) \leq c\}$ such that $(R_{g\mathcal{G}}(e))^\top R_{g\mathcal{G}}(e)$ is positive definite for any $e \in \Omega_c$ and $(R_{g\mathcal{G}}(e))^\top \tilde{d} \neq 0$ for any $e \in \Omega_c$ and $e \notin E_e$.

Proof.

First, due to the infinitesimal rigidity of $(\mathcal{G}, p^*)$, if a point p is sufficiently close to E_p, then $(\mathcal{G}, p)$ is infinitesimally rigid, which, together with *Lemma 12*, implies that $(R_{g\mathcal{G}}(p))^\top R_{g\mathcal{G}}(p)$ is positive definite. Thus there exists a positive constant ρ_{max} such that if $\rho_{max} \geq \rho > 0$, then $(R_{g\mathcal{G}}(e))^\top R_{g\mathcal{G}}(e)$ is positive definite for any $e \in \Omega_\rho = \{e : V(e) \leq \rho\}$. □

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Proof.

(Cont.) Second, since $\phi(p)$, which is the potential function, is a real analytic function in some neighborhood of any $\bar{p} \in E_p$, it follows from *Theorem 2* that there exist a neighborhood $\mathcal{U}_{\bar{p}}$ of $\bar{p}$ and constants $k_{\bar{p}} > 0$ and $\rho_{\bar{p}} \in [0, 1)$ such that

$$\|\nabla\phi(p)\| = \| - k_g(R_{g_\mathcal{G}}(p))^\top \tilde{d}\| \geq k_{\bar{p}}\|\phi(p) - \phi(\bar{p})\|^{\rho_{\bar{p}}}$$

for all $p \in \mathcal{U}_{\bar{p}}$. Since $\phi(p) = 0$ only if $p \in E_p$,

$$\|k_g(R_{g_\mathcal{G}}(p))^\top \tilde{d}\| \geq k_{\bar{p}}\|\phi(p)\|^{\rho_{\bar{p}}} > 0 \tag{6}$$

for all $p \in \mathcal{U}_{\bar{p}}$ and $p \notin E_p$. Then, for any $\bar{e} \in E_e$, we can take a neighborhood $\mathcal{U}_{\bar{e}}$ of $\bar{e}$ such that

$$\|(R_{g_\mathcal{G}}(e))^\top \tilde{d}\| > 0$$

for all $e \in \mathcal{U}_{\bar{e}}$ and $e \notin E_e$. □

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$$\|k_g (R_{g_\mathcal{G}}(p))^\mathsf{T} \tilde{d}\| \geq k_{\bar{p}} \|\phi(p)\|^{\rho_{\bar{p}}} > 0 \tag{7}$$

for all $p \in \mathcal{U}_{\bar{p}}$ and $p \notin E_p$. Then, for any $\bar{e} \in E_e$, we can take a neighborhood $\mathcal{U}_{\bar{e}}$ of $\bar{e}$ such that

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for all $p \in \mathcal{U}_{\bar{p}}$. Since $\phi(p) = 0$ only if $p \in E_p$,

$$\|k_g(R_{g_\mathcal{G}}(p))^\mathsf{T}\tilde{d}\| \geq k_{\bar{p}}\|\phi(p)\|^{\rho_{\bar{p}}} > 0 \quad (8)$$

for all $p \in \mathcal{U}_{\bar{p}}$ and $p \notin E_p$. Then, for any $\bar{e} \in E_e$, we can take a neighborhood $\mathcal{U}_{\bar{e}}$ of $\bar{e}$ such that

$$\|(R_{g_\mathcal{G}}(e))^\mathsf{T}\tilde{d}\| > 0 \quad (9)$$

for all $e \in \mathcal{U}_{\bar{e}}$ and $e \notin E_e$. □

Proof.

(Cont.) Third, due to the compactness of E_e, there exists a finite open cover $\mathcal{U}_{E_e} = \bigcup_{k=1}^{n_e} \mathcal{U}_{\bar{e}_k}$ such that (9) holds for all $e \in \mathcal{U}_{E_e}$ and $e \notin E_e$. That is, for any $k \in \{1, \ldots, n_e\}$, if $e \in \mathcal{U}_{\bar{e}_k}$ and $e \notin E_e$, then (9) holds. Taking $\mathcal{U}_{E_e}$ and c such that $\Omega_c \subseteq \mathcal{U}_{E_e}$ and $c \leq \rho_{max}$ ensures that $(R_{g_\mathcal{G}}(e))^\top R_{g_\mathcal{G}}(e)$ is positive definite for any $e \in \Omega_c$ and $(R_{g_\mathcal{G}}(e))^\top \tilde{d} \neq 0$ for any $e \in \Omega_c$ and $e \notin E_e$. □

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General case

Theorem

For N-agents, if $(\mathcal{G}, p^*)$ is infinitesimally rigid in the plane, E_{p^*} is locally asymptotically stable under the proposed control law.

Proof.

Take $V(e) = (k_d/4)\sum_{(i,j)\in\mathcal{E}}(\|e_{ij}\|^2 - d_{ij}^*)^2$ as a Lyapunov function. The time derivative of $V(e)$ is then arranged as

$$\dot{V}(e) = -k_d\tilde{d}^\top R_{g\mathcal{G}}(e)((R_{g\mathcal{G}}(e))^\top R_{g\mathcal{G}}(e))^{-1}(R_{g\mathcal{G}}(e))^\top\tilde{d}.$$

From *Lemma 14*, there exists a level set Ω_c such that $(R_{g\mathcal{G}}(e))^\top R_{g\mathcal{G}}(e)$ is positive definite for any $e \in \Omega_c$ and $(R_{g\mathcal{G}}(e))^\top\tilde{d} \neq 0$ for any $e \in \Omega_c$ and $e \notin E_e$. Since $\dot{V}(e)$ is negative definite in Ω_c, E_e is locally asymptotically stable, which in turn implies the local asymptotic stability of E_{p^*}. □

General case

Theorem

Given an N-agent group, if $(\mathcal{G}, p^*)$ is infinitesimally rigid, then the control law (5) achieves the asymptotic convergence of p to a point in E_p.

Proof.

From *Lemma 14*, there exists a level set Ω_c such that $(R_{g\mathcal{G}}(e))^\mathsf{T} R_{g\mathcal{G}}(e)$ is positive definite for any $e \in \Omega_c$ and $(R_{g\mathcal{G}}(e))^\mathsf{T}\tilde{d} \neq 0$ for any $e \in \Omega_c$ and $e \notin E_e$. Since $((R_{g\mathcal{G}}(e))^\mathsf{T} R_{g\mathcal{G}}(e))^{-1}$ is positive definite in Ω_c, there exists a constant M_R such that $\|((R_{g\mathcal{G}}(e))^\mathsf{T} R_{g\mathcal{G}}(e))^{-1}\|_1 \leq M_R$, where $\|\cdot\|_1$ denotes the induced 1-norm of matrices. It can be followed by using the result from [Krick et al. -2009, IJC] that $u(t) = -(k_d/4k_g)((R_{g\mathcal{G}}(e))^\mathsf{T} R_{g\mathcal{G}}(e))^{-1} u_g(t)$ also belongs to $\mathcal{L}_1$ space. Thus p asymptotically converges to a point in E_p. □

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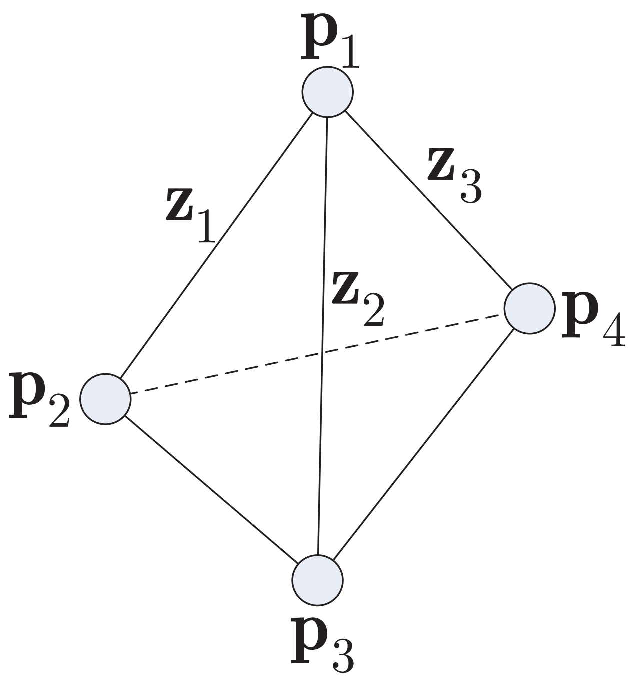
Notation

- Relative displacements: $\mathbf{z}_1 = \mathbf{p}_2 - \mathbf{p}_1$, $\mathbf{z}_2 = \mathbf{p}_3 - \mathbf{p}_1$, $\mathbf{z}_3 = \mathbf{p}_4 - \mathbf{p}_1$
- A square matrix: $Z = \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \end{bmatrix}$.
- Remark that $\frac{1}{2}|\det Z|$ is the volume of the tetrahedron in the figure.
- Squared-distance error:

$$e_{ij}(t) = \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|^2 - d_{ij}^2, \quad \forall(i,j) \in \mathcal{E},$$

  where $d_{ij} = \|\bar{\mathbf{p}}_i - \bar{\mathbf{p}}_j\|$, $\forall(i,j) \in \mathcal{E}$.
  Define $\mathbf{e} = \begin{bmatrix} e_{12} & \dots & e_{34} \end{bmatrix}^\mathsf{T}$.
- Assumption for simplification: $d_{ij} = d > 0$, $\forall(i,j) \in \mathcal{E}$
  $\Rightarrow$ a regular tetrahedron shape.



Gradient-descent law

- A potential function: $\phi(\mathbf{p}) = \frac{1}{4}\mathbf{e}^{\mathsf{T}}\mathbf{e} = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} e_{ij}^2$.
- Objective: $\lim_{t\to\infty} \phi(\mathbf{p}(t)) = 0$,
  $\lim_{t\to\infty} \mathbf{p}(t) =$ a finite point.
- Gradient-descent law:

$$\mathbf{u} = -\nabla\phi = = -\left[\frac{\partial \phi}{\partial \mathbf{p}}\right]^{\mathsf{T}} = -R_{\mathcal{G}}^{\mathsf{T}}\mathbf{e} \tag{10}$$

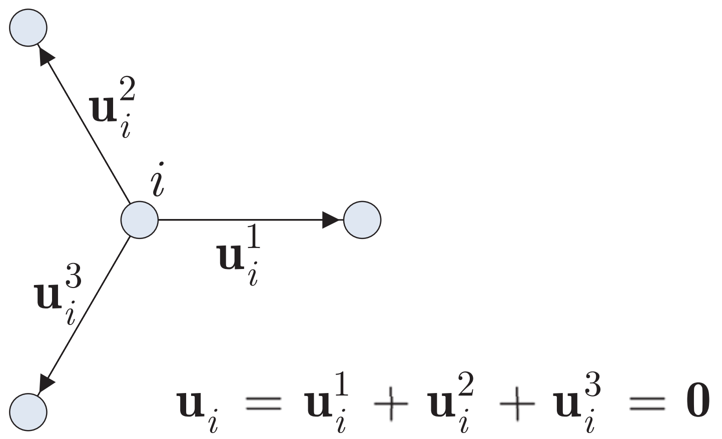
$$= \begin{bmatrix} e_{12}\mathbf{z}_1 + e_{13}\mathbf{z}_2 + e_{14}\mathbf{z}_3 \\ (-e_{12} - e_{23} - e_{24})\mathbf{z}_1 + e_{23}\mathbf{z}_2 + e_{24}\mathbf{z}_3 \\ e_{23}\mathbf{z}_1 + (-e_{13} - e_{23} - e_{34})\mathbf{z}_2 + e_{34}\mathbf{z}_3 \\ e_{24}\mathbf{z}_1 + e_{34}\mathbf{z}_2 + (-e_{14} - e_{24} - e_{34})\mathbf{z}_3 \end{bmatrix}, \tag{11}$$

$\Leftrightarrow \forall i \in \mathcal{V}$, $\mathbf{u}_i = \sum_{j\in\mathcal{N}_i}(\|\mathbf{p}_j - \mathbf{p}_i\|^2 - d^2)(\mathbf{p}_j - \mathbf{p}_i)$, where $\mathcal{N}_j$ is the set of neighbors of i.

Equilibrium states

$$\mathbf{u}_i = \sum_{j\in\mathcal{N}_i} (\|\mathbf{p}_j - \mathbf{p}_i\|^2 - d^2)(\mathbf{p}_j - \mathbf{p}_i) \triangleq \mathbf{u}_i^1 + \mathbf{u}_i^2 + \mathbf{u}_i^3.$$

- Desired equilibrium state: $\|\mathbf{p}_j - \mathbf{p}_i\| = d, \forall (i,j) \in \mathcal{E}$.
- Undesired equilibrium state: $\exists (i,j) \in \mathcal{E}, \|\mathbf{p}_j - \mathbf{p}_i\| \neq d, \forall k \in \mathcal{V}, \mathbf{u}_k = \mathbf{0}$.



Some sets

- Equilibrium sets:

$$\mathcal{Q} = \left\{\mathbf{p} \in \mathbb{R}^{3|\mathcal{V}|} : \nabla\phi = \mathbf{0}\right\},$$

$$\mathcal{D} = \left\{\mathbf{p} \in \mathbb{R}^{3|\mathcal{V}|} : \mathbf{e} = \mathbf{0}\right\},$$

$$\mathcal{U} = \mathcal{Q} \setminus \mathcal{D} = \left\{\mathbf{p} \in \mathbb{R}^{3|\mathcal{V}|} : \nabla\phi = \mathbf{0},\ \mathbf{e} \neq \mathbf{0}\right\}.$$

- A set by collinear agents:

$$\mathcal{C} = \left\{\mathbf{p} \in \mathbb{R}^{3|\mathcal{V}|} :\ \det Z = 0\right\}.$$

$\Rightarrow$ all agents exist on a plane.

- Analysis on ϕ:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial\mathbf{p}}\dot{\mathbf{p}} = -\|\nabla\phi\|^2 \leq 0, \quad \because \dot{\mathbf{p}} = \mathbf{u} = -\nabla\phi.$$

$\Rightarrow \lim_{t\to\infty} \nabla\phi = \mathbf{0} \Rightarrow \mathbf{p}(t)$ approaches $\mathcal{Q}(= \mathcal{D} \cup \mathcal{U})$.

Attractiveness of the equilibrium sets

- Two cases:

$$\lim_{t\to\infty} \nabla\phi = \mathbf{0} \quad \text{and} \quad \begin{cases} \lim_{t\to\infty} \mathbf{e} = \mathbf{0} & \Rightarrow & \mathbf{p}(t) \text{ approaches } \mathcal{D}. \\ \qquad\text{or} & & \\ \lim_{t\to\infty} \mathbf{e} \neq \mathbf{0} & \Rightarrow & \mathbf{p}(t) \text{ approaches } \mathcal{U}. \end{cases}$$

- Note that $\dot{\phi}$ is zero if and only if $\mathbf{u} = \mathbf{0}$.
- Since $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ exist in $\mathbb{R}^3$, if they are linearly independent, then $\dot{\mathbf{p}} = \mathbf{0}$ implies that $\mathbf{e} = \mathbf{0}$ from (11).

Lemma

If $\mathbf{p} \in \mathcal{U}$, then $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ are linearly dependent.

- Lemma 18 means that any formation, with $\mathbf{e} \neq \mathbf{0}$, formed by $\forall \mathbf{p} \in \mathcal{U}$ should exist on a plane due to the linear dependence of $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$, which means that $\det Z = 0$. Hence, $\mathcal{U} \subset \mathcal{C}$.

Repulsiveness of the undesired equilibrium set

- We have
$$\frac{\mathrm{d}}{\mathrm{d}t}\det Z = -2\sigma \det Z \quad \Rightarrow \quad \det Z = \exp\left[-2\int_0^t \sigma(\mathbf{p}(s))\,\mathrm{d}s\right]\det Z_0,$$
where $\det Z_0$ is $\det Z$ at $t = 0$, and $\sigma = \sum_{(i,j)\in\mathcal{E}} e_{ij}$.
- If $\mathcal{U}$ is attractive, then $\det Z$ converges to 0 because $\mathcal{U} \subseteq \mathcal{C}$.

$$\det Z = \underbrace{\exp\left[-2\int_0^t \sigma(\mathbf{p}(s))\,\mathrm{d}s\right]}_{>0}\det Z_0,$$

- It is true that $\det Z \neq 0$ for all $t \geq 0$ if and only if $\det Z_0 \neq 0$.
- There is a neighborhood of $\mathcal{U}$ in which $\sigma < 0$ for all $\mathbf{p}$.
- $\exp[\cdot]$ does not converges to zero, which contradicts to the hypothesis.

Lemma

If $\mathbf{p}(0) \notin \mathcal{C}$, then $\mathbf{p}(t)$ is bounded away from $\mathcal{U}$ for all $t \geq 0$.

Main theorem

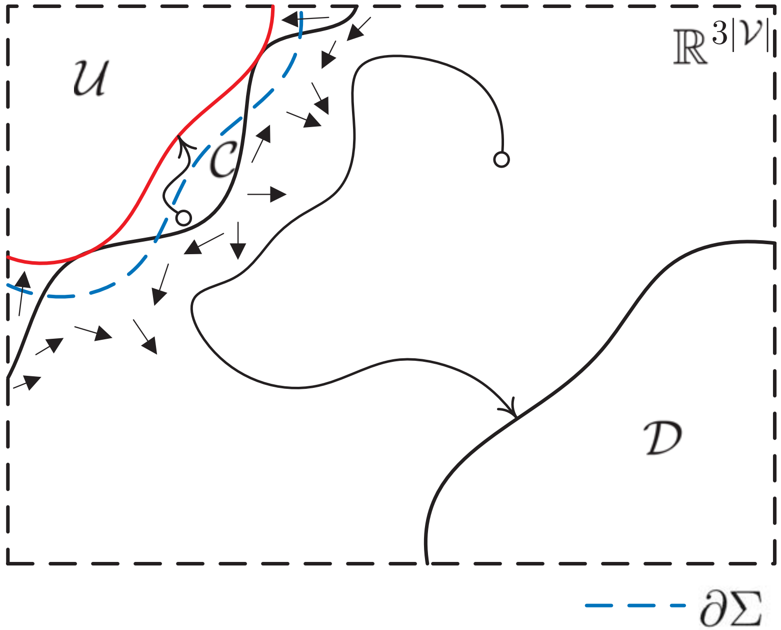
Theorem

For a given regular tetrahedral formation $(\mathcal{G}, \bar{\mathbf{p}})$ and the gradient-descent law, the realization $\mathbf{p}(t)$ converges to a finite point which is congruent to $\bar{\mathbf{p}}$ if and only if the initial condition $\mathbf{p}(0)$ satisfies $\mathbf{p}(0) \notin \mathcal{C}$.

Corollary

The realization $\mathbf{p}(t)$ approaches $\mathcal{U}$ if $\mathbf{p}(0) \in \mathcal{C}$.

- Σ: a neighborhood of $\mathcal{U}$.
- $\partial\Sigma$: the boundary of Σ.



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Assumptions

- Quadratic potential function: $V(p) = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} e_{ij}^2$
- Gradient-descent law:

$$\dot{p} = -\left[\frac{\partial V}{\partial p}\right]^T = -[R(p)]^T e(p) = -(E(p)\otimes I_3)p. \tag{12}$$

- No mismatched desired distances.

$$R(p) \triangleq \frac{1}{2}\frac{\partial e}{\partial p} = \begin{bmatrix} p_1^T - p_2^T & p_2^T - p_1^T & 0 & 0 \\ p_1^T - p_3^T & 0 & p_3^T - p_1^T & 0 \\ p_1^T - p_4^T & 0 & 0 & p_4^T - p_1^T \\ 0 & p_2^T - p_3^T & p_3^T - p_2^T & 0 \\ 0 & p_2^T - p_4^T & 0 & p_4^T - p_2^T \\ 0 & 0 & p_3^T - p_4^T & p_4^T - p_3^T \end{bmatrix}. \tag{13}$$

- Further we assume that $R(\bar{p})$ has full row rank, which is equivalent that the framework $(\mathcal{K}_4, \bar{p})$ is rigid[1] (i.e., m = 3n-6).

[1] Rigorously say, infinitesimally rigid

Existing Result

- $p(t)$ approaches equilibrium set as $t \to \infty$.
- The origin of the error dynamics is (locally) exponentially stable.

$$\dot{V} = -eRR^T e \leq -4\left[\lambda_{\text{min}}(RR^T)\right] V \leq 0. \tag{14}$$

  - The matrix RR^T is positive definite near the desired formation shape from the assumption on $\bar{p}$. $\Rightarrow \lambda_{\text{min}}(RR^T) > 0$.

Analysis on Incorrect Equilibria

- Consider incorrect equilibrium set given by

$$\mathcal{P}_i = \left\{ p \in \mathbb{R}^{3|\mathcal{V}|} : \dot{p} = -[R(p)]^T e(p) = -(E(p) \otimes I_3) p = 0,\ e(p) \neq 0 \right\}. \tag{15}$$

- Linearization:

$$\frac{\partial}{\partial p}\left[-\frac{\partial V}{\partial p} \right]^T = -H_V(p), \tag{16}$$

where H_V is the Hessian matrix of V by definition.

- Investigate the existence of negative eigenvalue(s) of H_V.

Analysis on Incorrect Equilibria

- Let p^* be an element in the incorrect equilibrium set. With an appropriate transformation, we have

$$\bar{H}_V(p^*) = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & E(p^*) \end{bmatrix}. \tag{17}$$

- Show the existence of negative eigenvalu(s) of $E(p^*)$, which can be done by finding a vector x such that $x^T[E(p^*) \otimes I_3]x < 0$.
- Actually, we have

$$x^T[E(p^*) \otimes I_3]x = -\sum_{(i,j)\in\mathcal{E}} [e_{ij}(p^*)]^2 < 0, \tag{18}$$

for $x = \bar{p}$.

Main Result I

Theorem

For almost every initial condition in $\mathbb{R}^{3|\mathcal{V}|}$, the trajectory $p(t)$ converges to the desired equilibrium set $\mathcal{P}_d$, where $\mathcal{P}_d = \left\{p \in \mathbb{R}^{3|\mathcal{V}|} : e(p) = 0\right\}$.

Proof.

By taking the derivative of V, we have $\dot{V} = \frac{\partial V}{\partial p}\dot{p} = -\left\|\frac{\partial V}{\partial p}\right\|^2 \leq 0$, which results in that r_{ij} and e_{ij} are bounded for all $i, j \in \mathcal{V}$. From the boundedness of r_{ij} and e_{ij}, we can also show that $\ddot{V}$ is bounded so $\dot{V}(p(t))$ is uniformly continuous in t on $[t_0, \infty)$ with an initial time t_0. Since $V(p(t))$ is a non-increasing lower bounded function, the limit of $V(p(t))$ exists. Therefore, $\dot{V}(p(t))$ converges to 0 as $t \to \infty$ from Barbalat's lemma, which means that $p(t)$ approaches either $\mathcal{P}_d$ or $\mathcal{P}_i$. The instability of the incorrect equilibrium set $\mathcal{P}_i$ has been previously shown. Therefore, for almost every initial condition in $\mathbb{R}^{3|\mathcal{V}|}$, $p(t)$ approaches $\mathcal{P}_d$. □

Main Result II

- Consider $Z(p) = [r_{12} \;\; r_{13} \;\; r_{14}] \in \mathbb{R}^{3\times 3}$, where $r_{ij} = p_i - p_j$. Let $\Delta(p(t)) = \det Z(p(t))$
- $|\Delta|$ is proportional to the volume occupied by the tetrahedron in $\mathbb{R}^3$.
- We can show that $\Delta(p^*) = 0$ for any p^* in the incorrect equilibrium set.
- Suppose that $(\mathcal{K}_4, p^*)$ is a point formation or has a planner shape.
  - We can show that $\Delta(p(t))$ cannot converge to 0 if $p(0)$ is not in $\mathcal{C} = \{p \in \mathbb{R}^{3|\mathcal{V}|} : \Delta(p) = 0\}$.
- Suppose that $(\mathcal{K}_4, p^*)$ is a line formation.
  - We can show that $p(t)$ is able to converge to p^* only if $(\mathcal{K}_4, p)$ is a line formation.

Corollary

The region of attraction for the desired equilibrium set $\mathcal{P}_d$ is $\mathbb{R}^{3|\mathcal{V}|} \setminus \mathcal{C}$.

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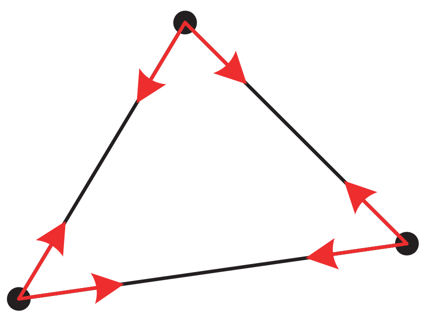
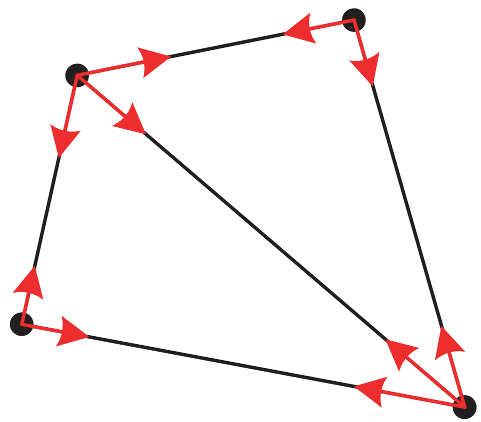
Rigid graphs in 2-D

- We have solved $K4$ in 3-D; but can we extend the results in 3-D into 2-D?
- Minimally infinitesimal rigid graph with four agents into 2-D?
- General minimally rigid graph in 2-D?

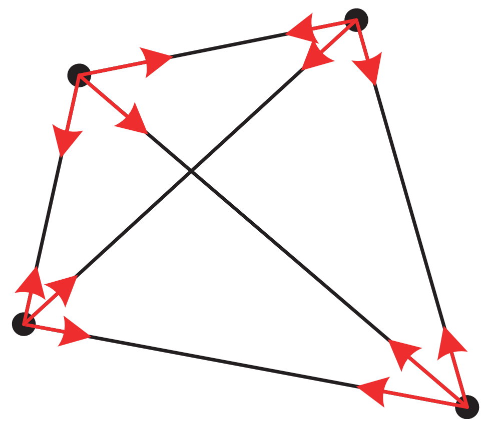
Rigid graphs in 2-D



Solved



Not solved (partially)



K4: Not solved

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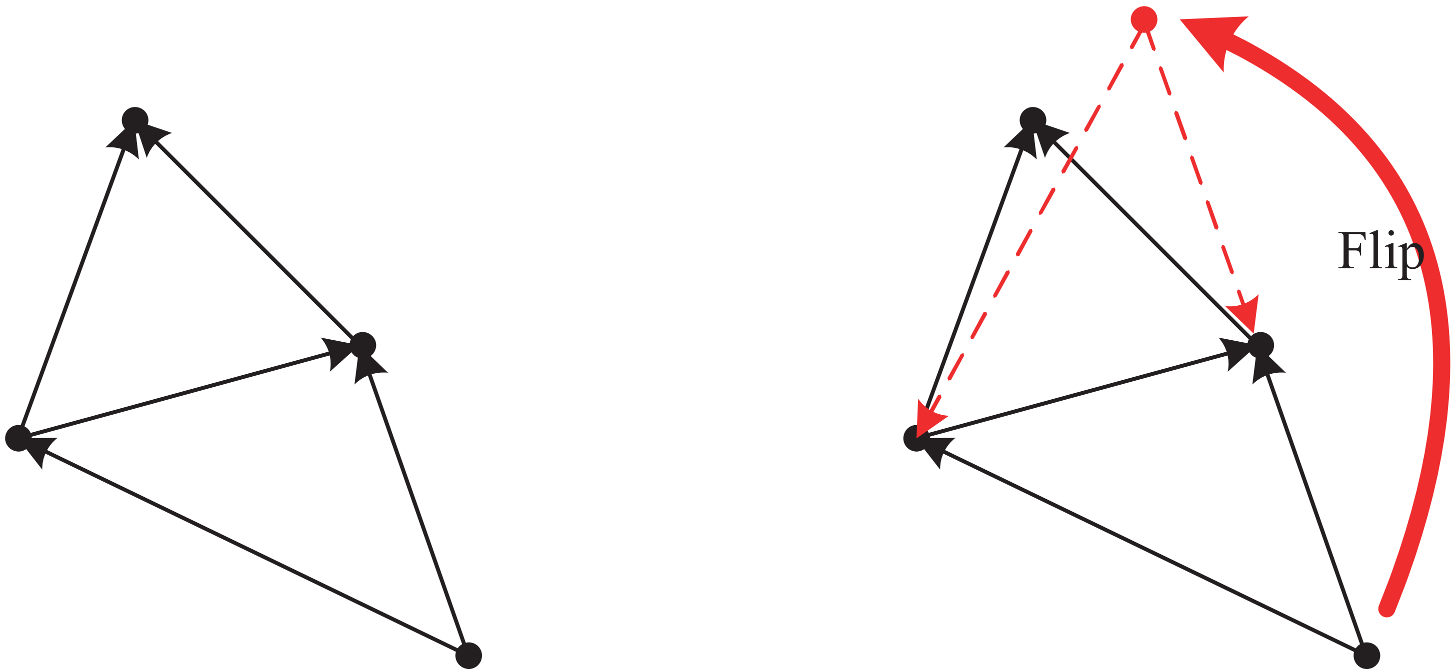
Persistence + global rigidity



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Thank You
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