presented by **Lars Grüne** in

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On the relation between dissipativity and the turnpike property

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supported by DFG Deutsche Forschungsgemeinschaft

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Outline

- Dissipativity and strict dissipativity
- The turnpike property and its variants
- Known results
- New results and proof ideas

System class

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x$$

with $x_{\mathbf{u}}(n) \in X$, $\mathbf{u}(n) \in U$, $\quad X$, U normed spaces

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Dissipativity and strict dissipativity

Dissipativity

$$x^+ = f(x, u)$$

Introduce functions $s : X \times U \to \mathbb{R}$ and $\lambda : X \to \mathbb{R}_0^+$

$s(x, u) \in \mathbb{R}$	supply rate, measuring the (possibly negative) amount of energy supplied to the system via the input u in the next time step
$\lambda(x) \geq 0$	storage function, measuring the amount of energy stored inside the system when the system is in state x

Dissipativity

Definition [Willems '72] The system is called dissipative if for all $x \in X$, $u \in U$ the inequality

$$\lambda(x^+) \leq \lambda(x) + s(x,u)$$

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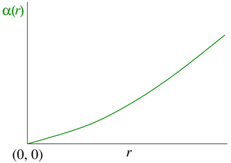
$\alpha \in \mathcal{K}$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$



Physical interpretation of dissipativity

$$\lambda(x^+) \le \lambda(x) + s(x,u) \;\; \big[-\alpha(\|x - x^e\|) \big]$$

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dissipativity: energy can only be dissipated (=lost) but not be generated inside the system

strict dissipativity: a certain amount of energy, depending on $\|x - x^e\|$ must be dissipated

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Translation to discrete time systems is quite straightforward

[Byrnes/Lin '94]

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Constructing F is particularly easy in case of passivity, because for $s(x, u) = \langle y, u \rangle$ it suffices to define the output feedback $F(y) := -y$

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Various stability properties can be formulated via dissipativity

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- (asymptotic) stability of the equilibrium x^e can be concluded for all solutions if the system is (strictly) dissipative, $s(x,u) \leq 0$ and the storage function λ is bounded from below and above by $\mathcal{K}_\infty$-functions in $\|x - x^e\|$

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In both cases, λ is a Lyapunov function

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Finally, strict dissipativity plays a major role in the analysis of so called economic model predictive control schemes (details later)

Available storage

Theorem [Willems '72, Byrnes/Lin '94] A system is [strictly] dissipative with supply rate s [and $\alpha \in \mathcal{K}$] if and only if

$$\sup_{K,\mathbf{u}} \sum_{k=0}^{K-1} -s(x_{\mathbf{u}}(k), \mathbf{u}(k)) \left[+ \alpha(\|x_{\mathbf{u}}(k) - x^e\|) \right] < \infty$$

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The particular storage function defined above is called

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We illustrate the property by two simple examples

Example 1: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x^+ = 2x + u$$

and spaces $X = [-2,2]$, $U = [-3,3]$

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$\rightsquigarrow$ optimal trajectory should stay near $x^e = 0$

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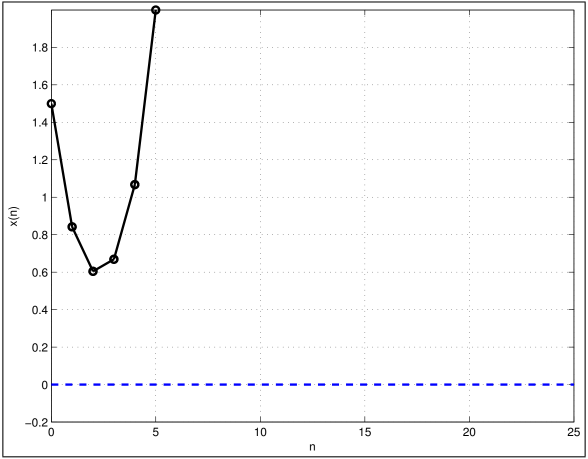

Optimal trajectory for $N = 5$

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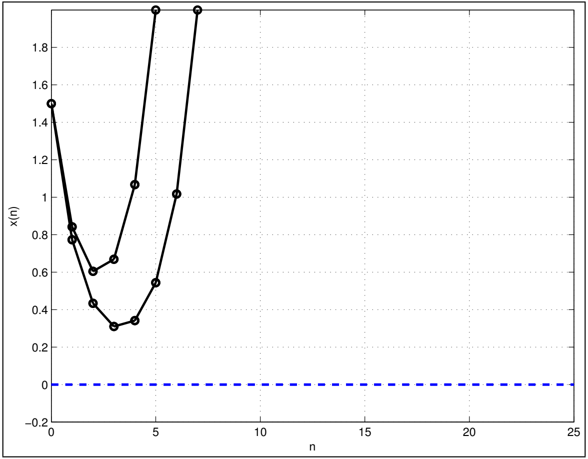

Optimal trajectories for $N = 5, \ldots, 7$

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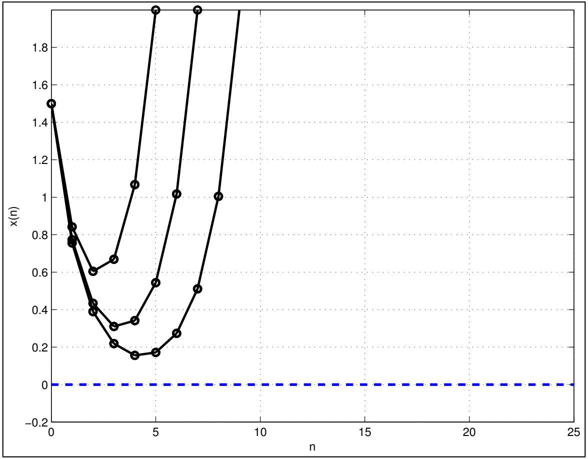

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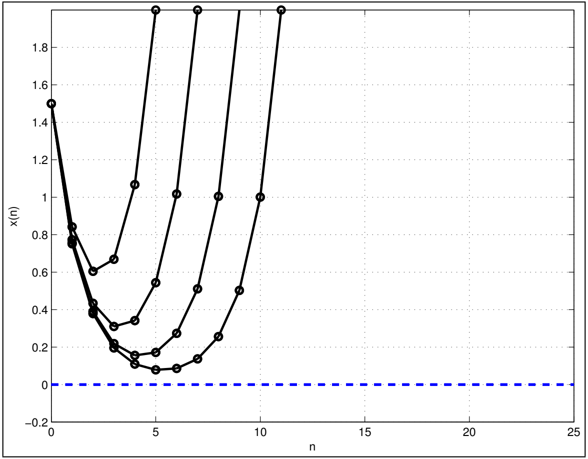

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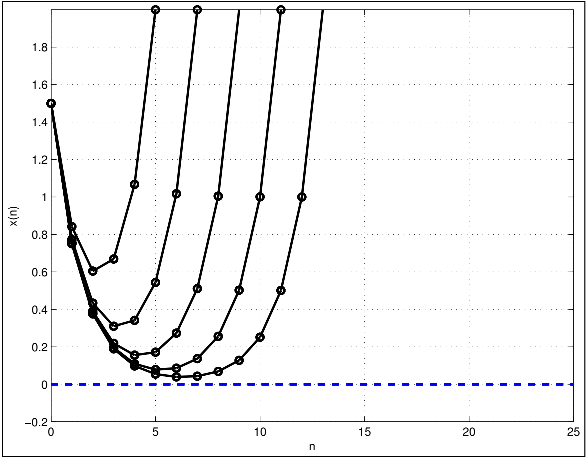

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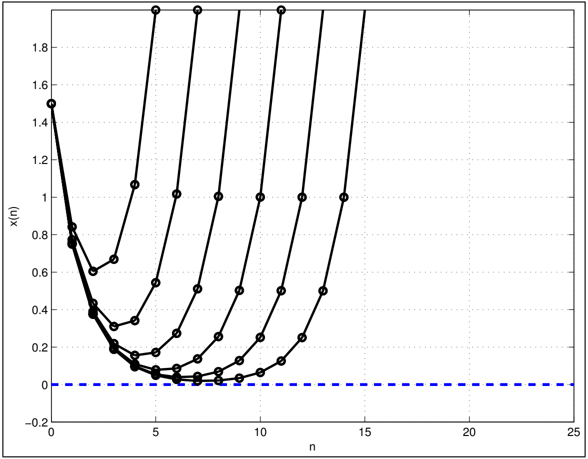

Optimal trajectories for $N = 5, \ldots, 15$

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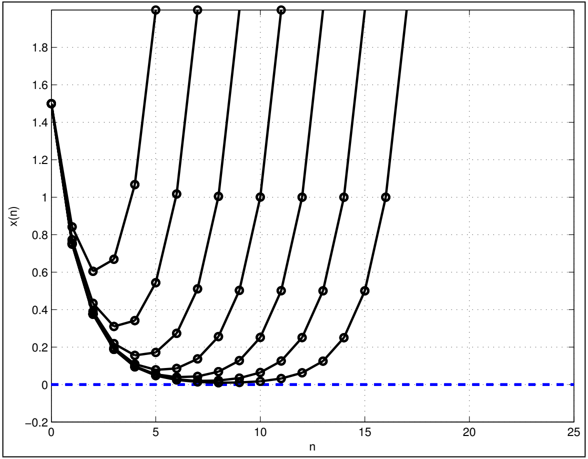

Optimal trajectories for $N = 5, \dots, 17$

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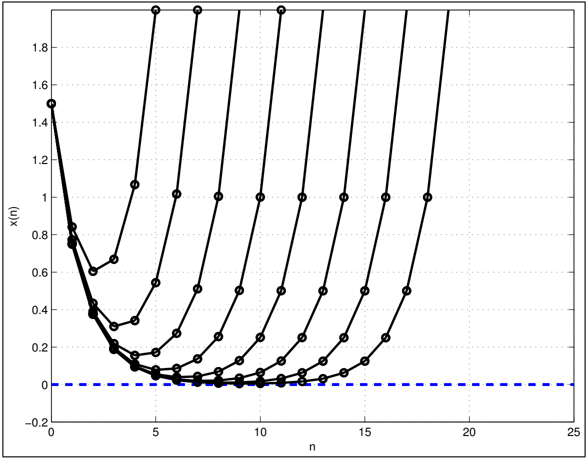

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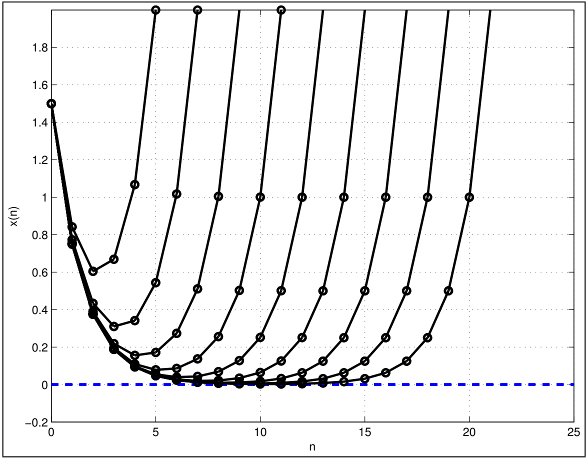

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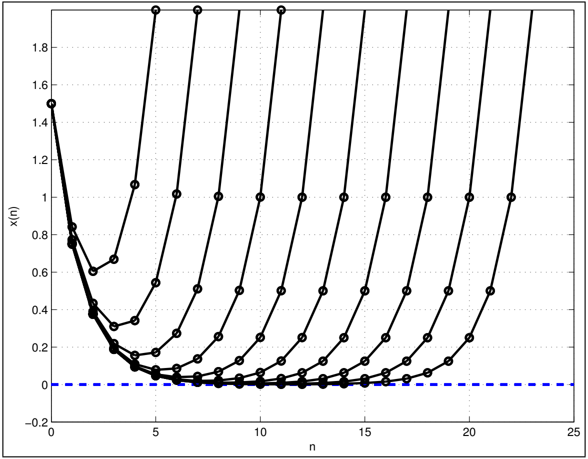

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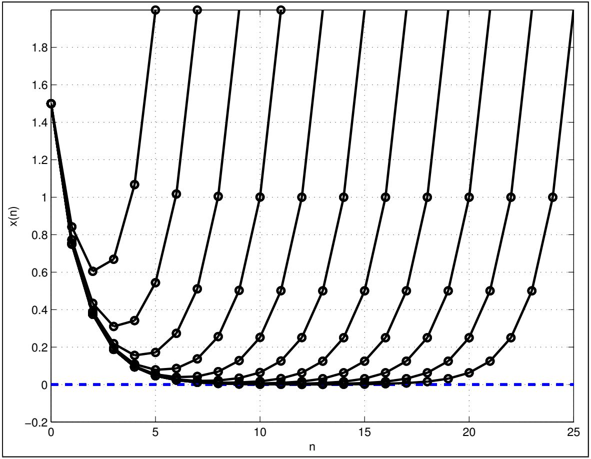

Optimal trajectories for $N = 5, \ldots, 25$

Example 2: a macroeconomic model

The second example is a 1d macroeconomic model

[Brock/Mirman '72]

Minimise the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $x^+ = u$

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On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

One may thus expect that finite horizon optimal trajectories also stay for a long time near that equilibrium

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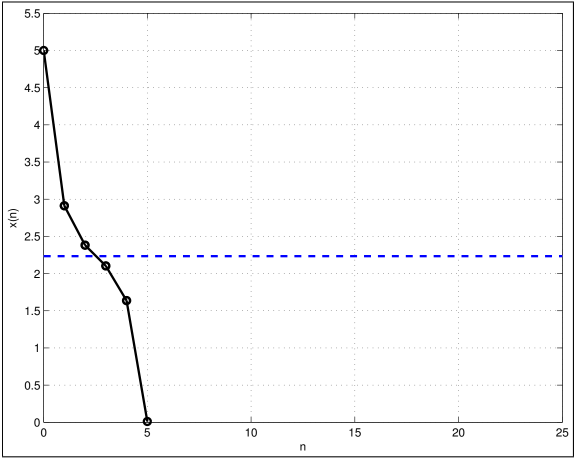

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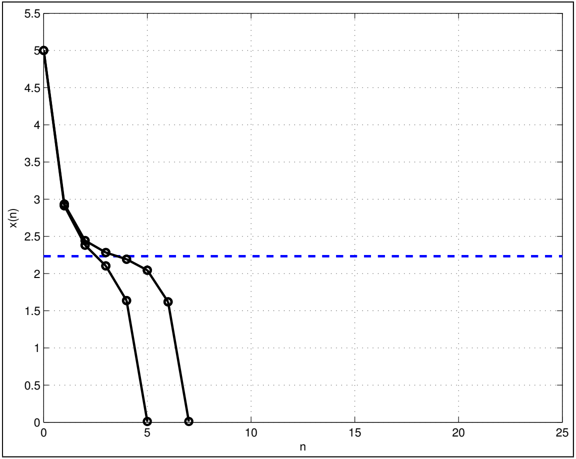

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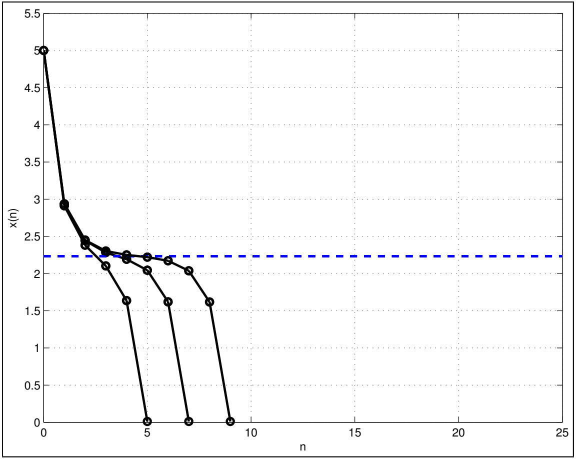

Optimal trajectories for $N = 5, \ldots, 9$

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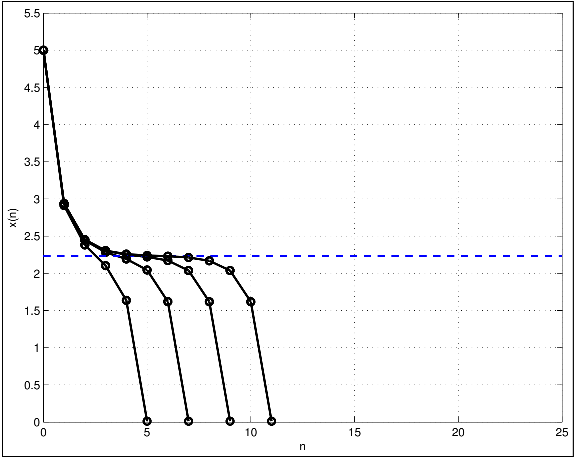

Optimal trajectories for $N = 5, \ldots, 11$

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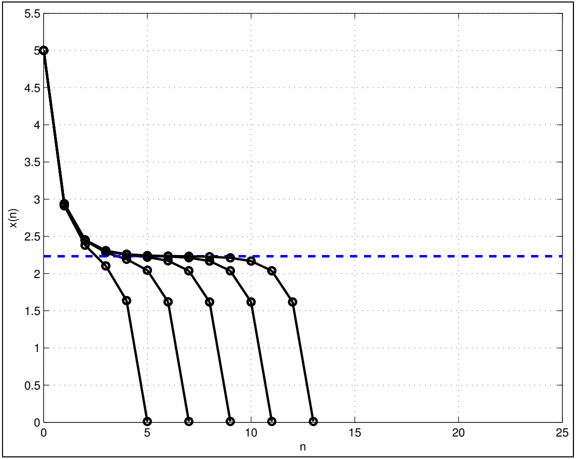
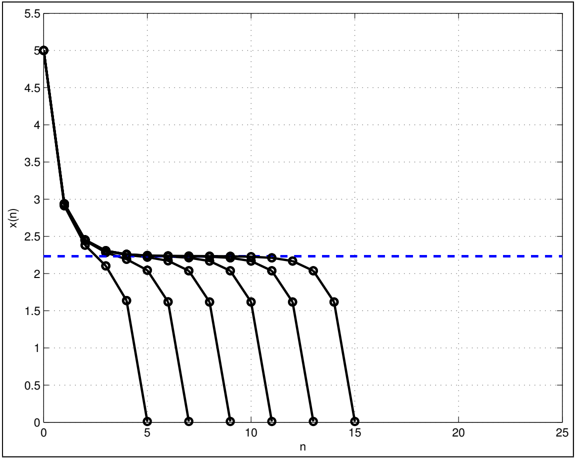
Example 2: optimal trajectories



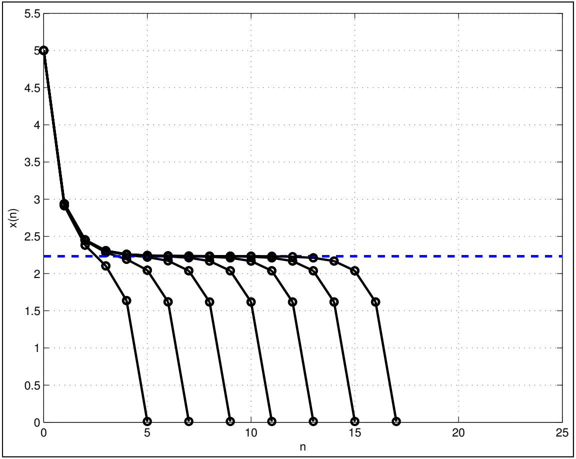
Optimal trajectories for $N = 5, \ldots, 13$

Example 2: optimal trajectories



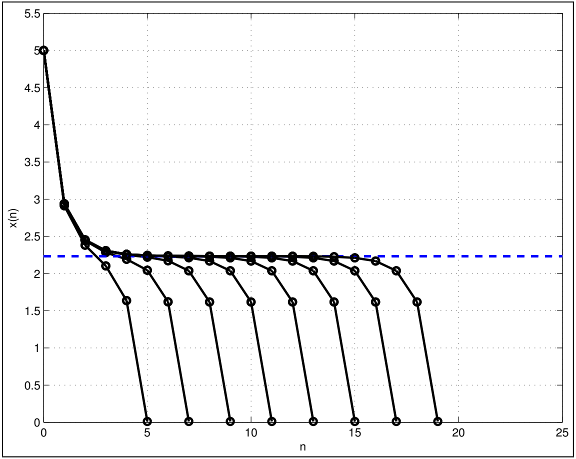
Optimal trajectories for $N = 5, \ldots, 15$

Example 2: optimal trajectories



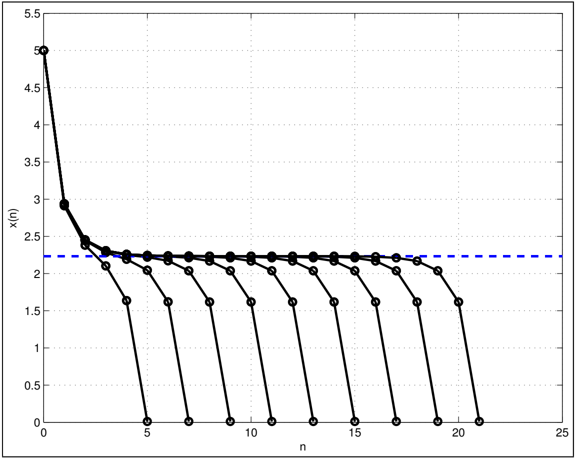
Optimal trajectories for $N = 5, \ldots, 17$

Example 2: optimal trajectories



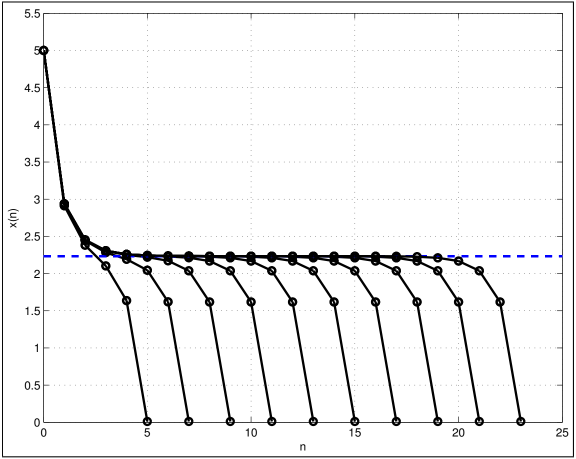
Optimal trajectories for $N = 5, \dots, 19$

Example 2: optimal trajectories



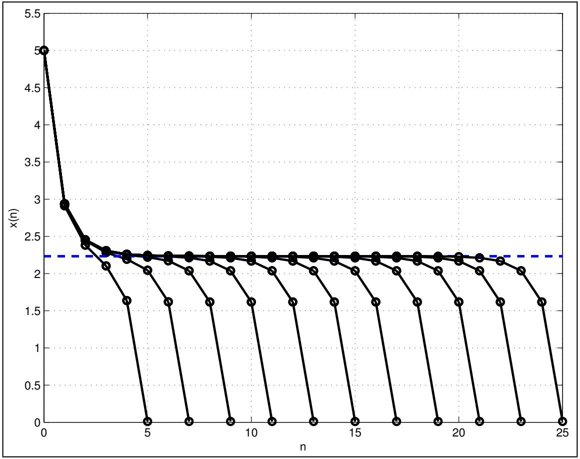
Optimal trajectories for $N = 5, \ldots, 21$

Example 2: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 23$

Example 2: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 25$

The turnpike property: formal definitions

Let x^e be an equilibrium, i.e., $f(x^e, u^e) = x^e$

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Turnpike property: There is $C > 0$ and $\rho \in \mathcal{K}_\infty$ such that for all $x \in X$ and $N \in \mathbb{N}$, all optimal trajectories $x^\star$ with $x^\star(0) = x$ and all $\varepsilon > 0$, the number

$$Q_\varepsilon := \#\{k \in \{0, \ldots, N-1\} \mid \|x^\star(k) - x^e\| \leq \varepsilon\}$$

satisfies $Q_\varepsilon \geq N - C/\rho(\varepsilon)$

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Near optimal turnpike property: There is $C > 0$ and $\rho \in \mathcal{K}_\infty$ such that for all $x \in X$, $N \in \mathbb{N}$ and $\delta > 0$, all trajectories $x_\mathbf{u}$ with $J_N(x, \mathbf{u}) \le V_N(x) + \delta$ and all $\varepsilon > 0$, the number

$$Q_\varepsilon := \#\{k \in \{0, \dots, N-1\} \,|\, \|x_\mathbf{u}(k) - x^e\| \le \varepsilon\}$$

satisfies $Q_\varepsilon \ge N - (C + \delta)/\rho(\varepsilon)$

History

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- Extensively studied in the 1970s in mathematical economy, cf. survey [McKenzie 1983]
- Renewed interest in recent years [Zaslavski '14, Trélat/Zuazua '15, Faulwasser et al. '15, ...]

Applications

Economists are interested in the turnpike property because it gives structural insight about optimal economic equilibria and the optimal trajectories' tendency to stay near them

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The finite horizon turnpike property at an equilibrium is also closely related to the convergence of infinite horizon optimal trajectories towards this equilibrium

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Ideas of this type can be found, e.g., in [Anderson/Kokotovic '87]

Application: Model predictive control

Turnpike properties are also pivotal for analysing economic Model Predictive Control (MPC) schemes

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MPC is a method in which an optimal control problem on an infinite horizon

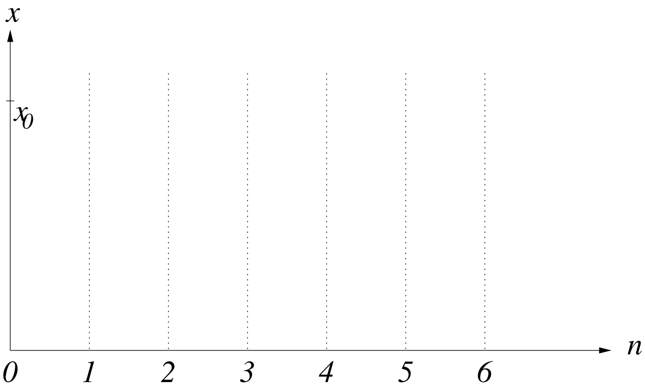
$$\text{minimise } J_\infty(x, \mathbf{u}) = \sum_{n=0}^{\infty} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n))$$

is approximated by the iterative solution of finite horizon problems

$$\text{minimise } J_N(x, \mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k))$$

with fixed $N \in \mathbb{N}$

MPC from the trajectory point of view



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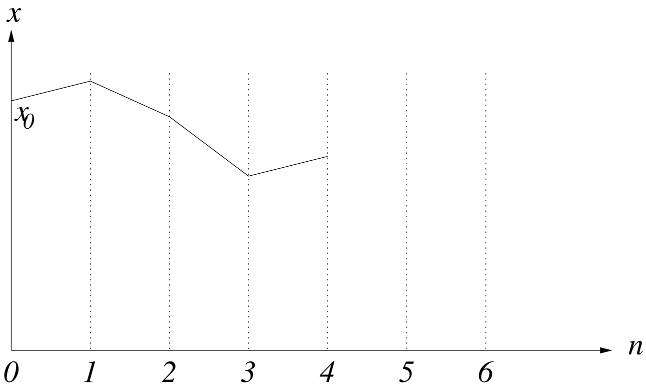
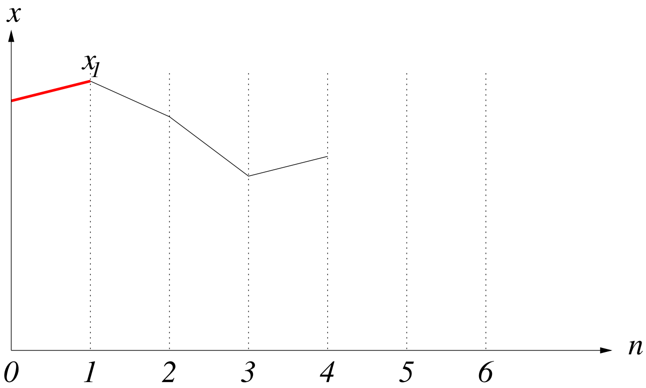


black = predictions (open loop optimization)

MPC from the trajectory point of view



black = predictions (open loop optimization)

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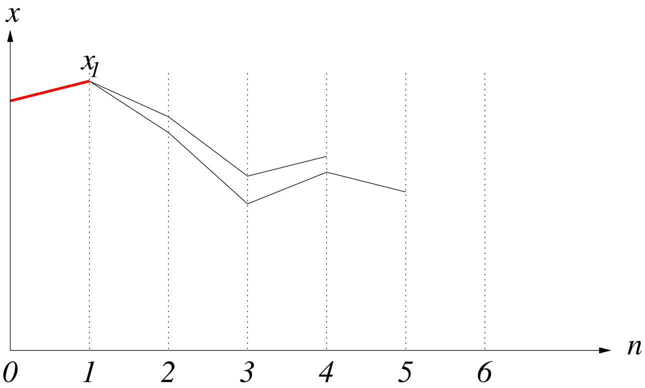

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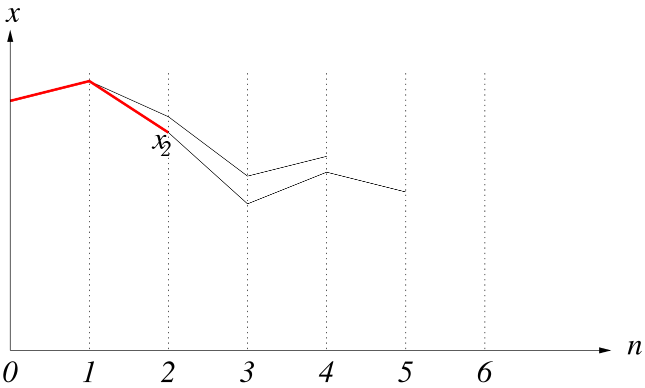

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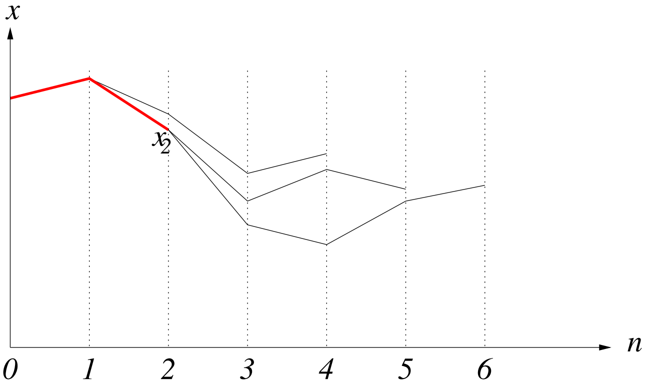

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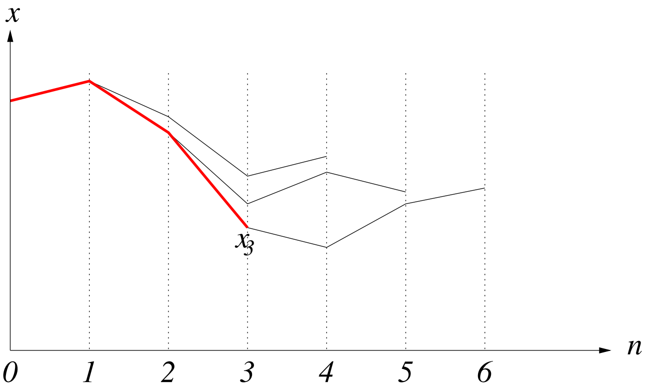

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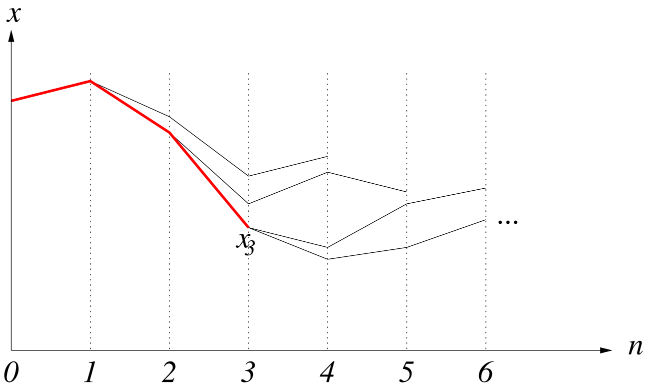

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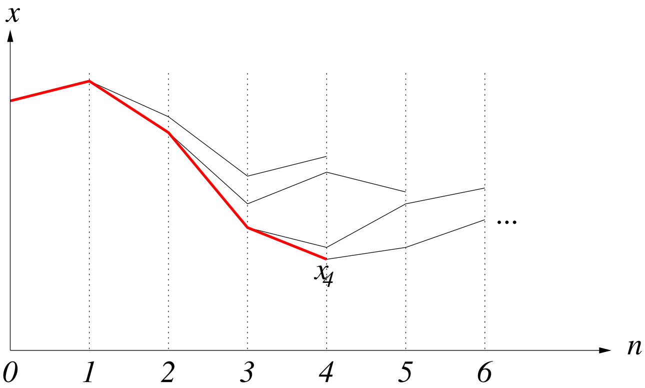

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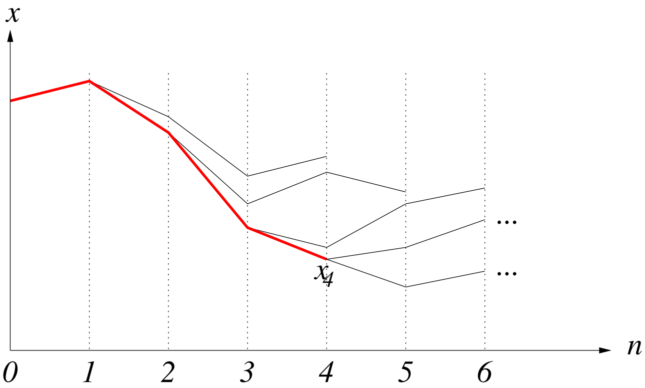

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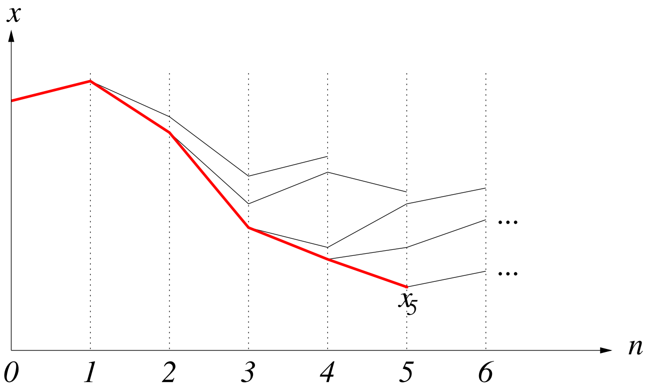

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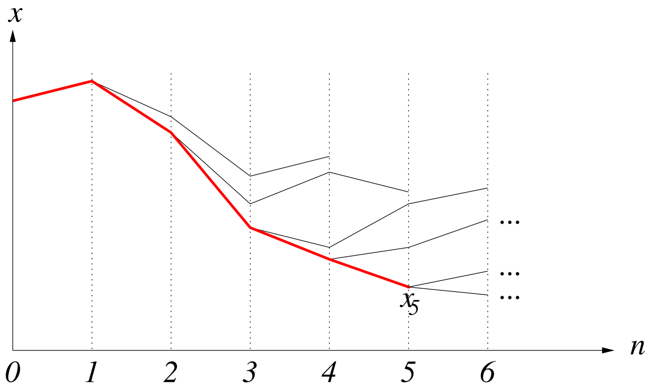

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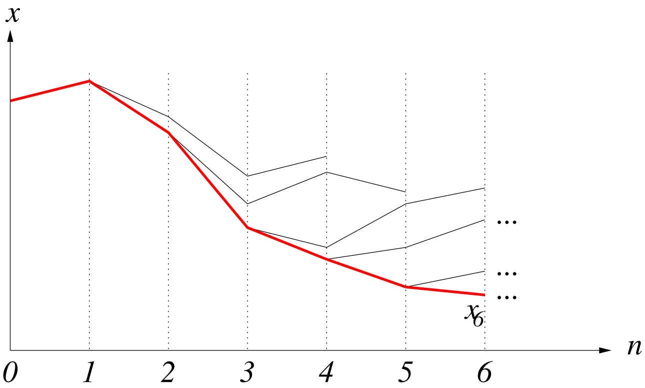

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Approximation result for MPC

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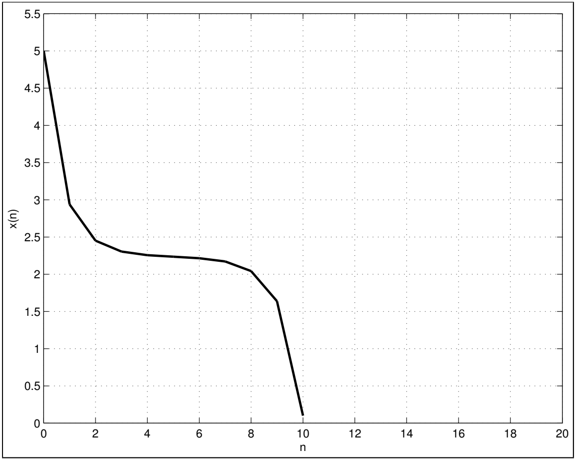
We illustrate this behaviour by our second example for $N = 10$

MPC for Example 2



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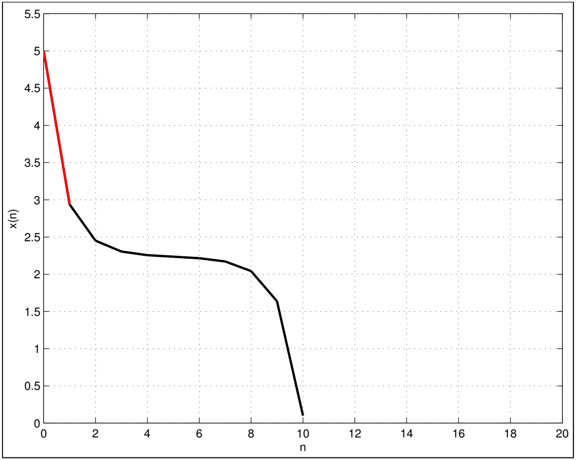

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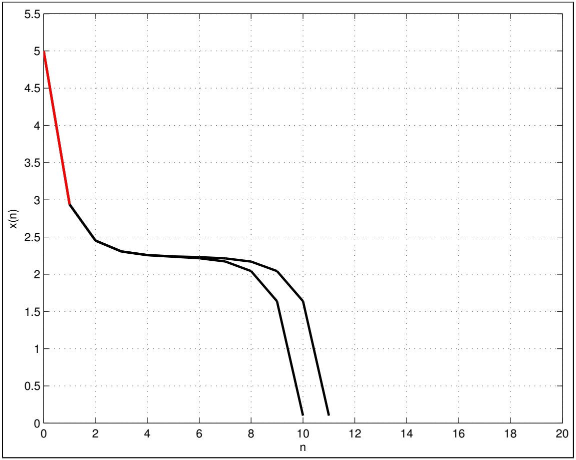

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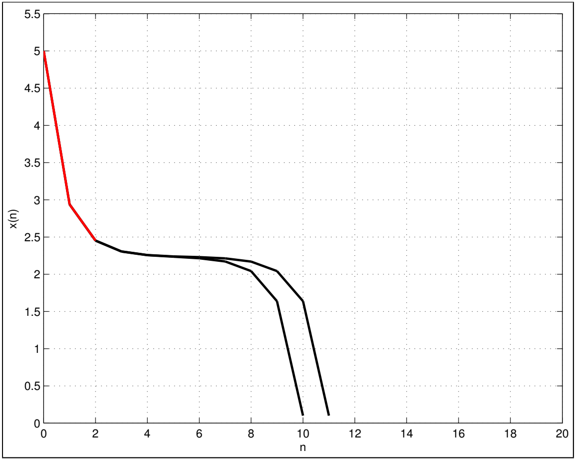

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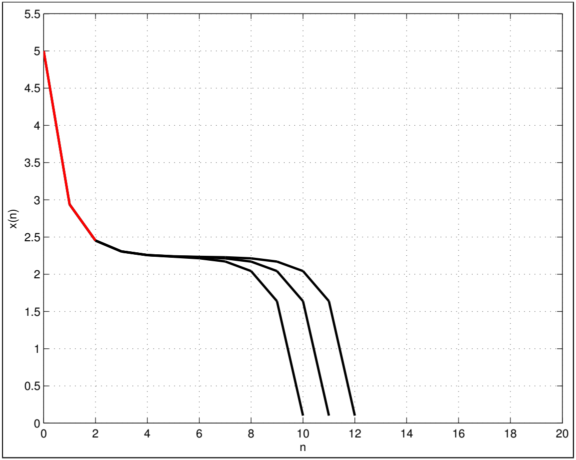

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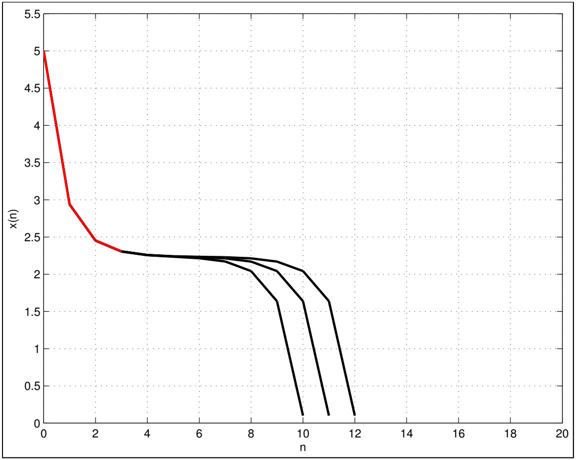

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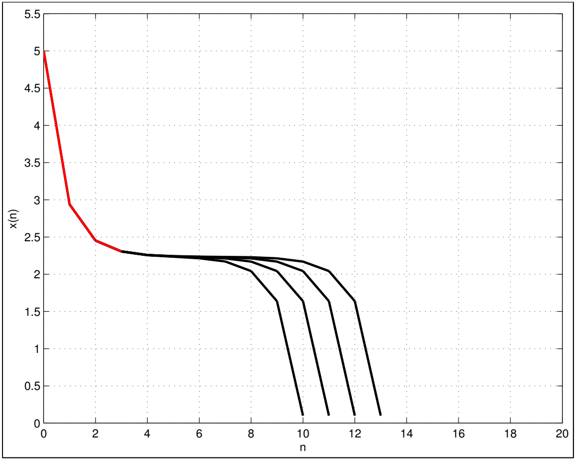

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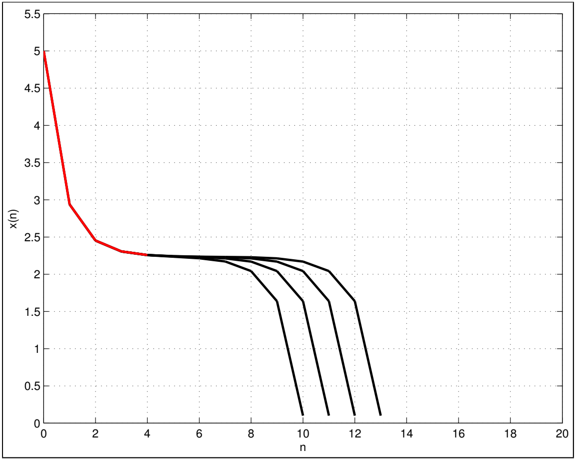

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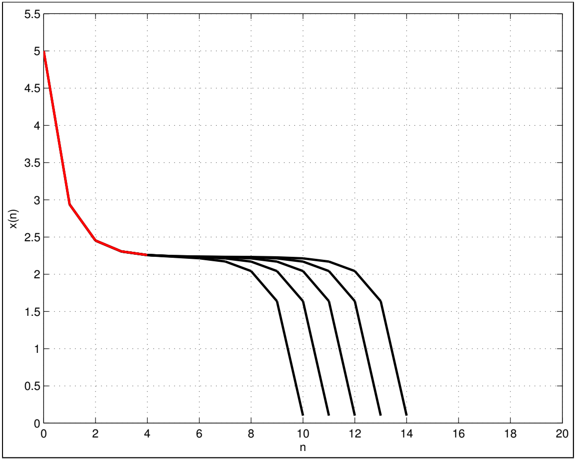

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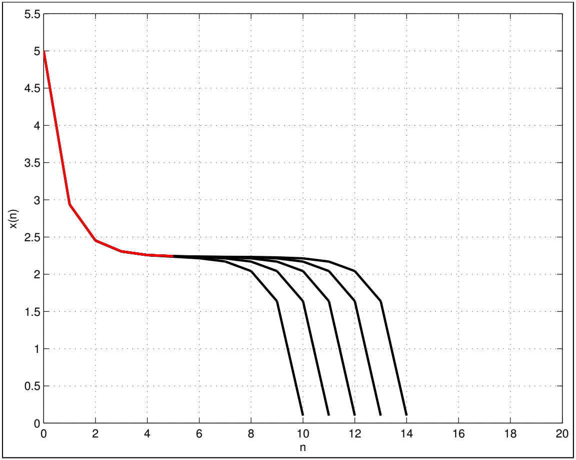

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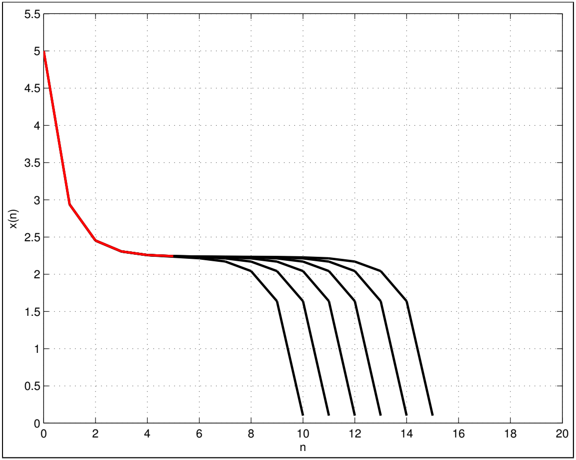

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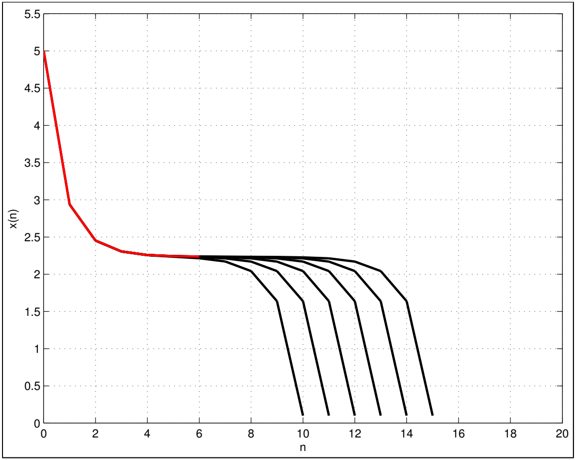

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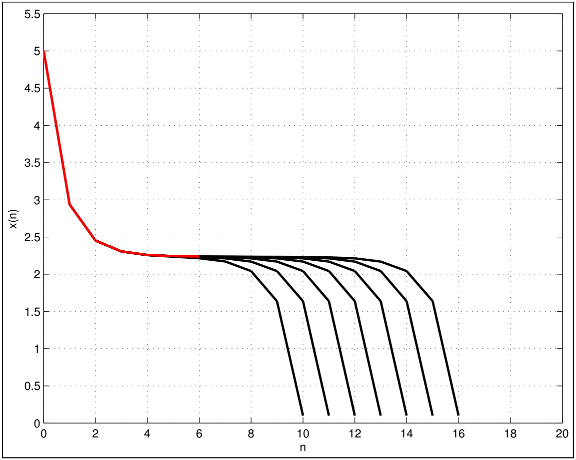

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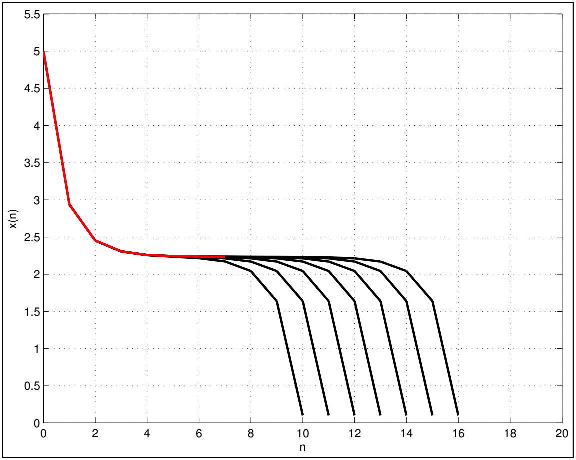

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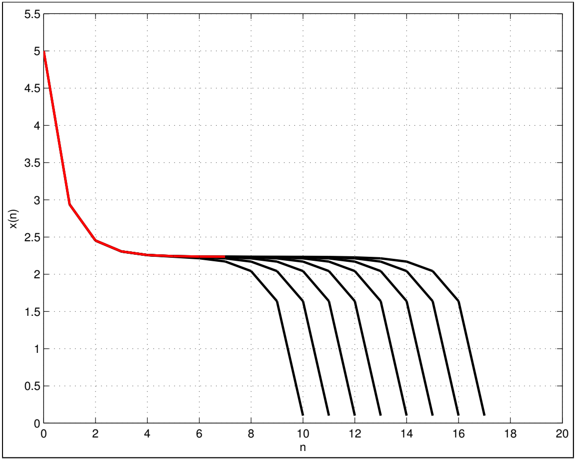

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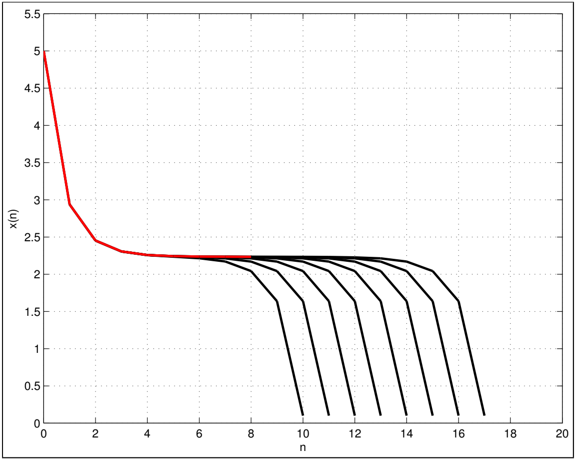

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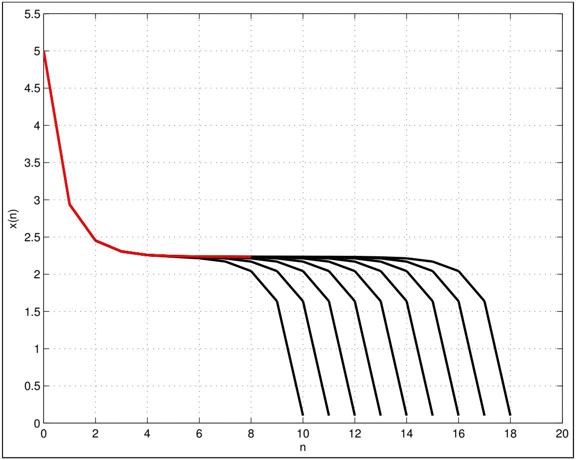

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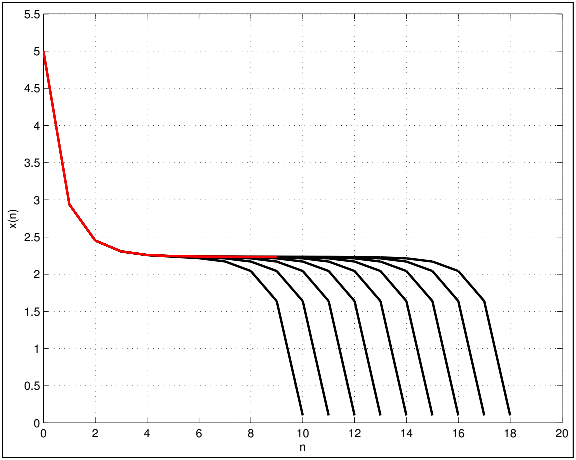

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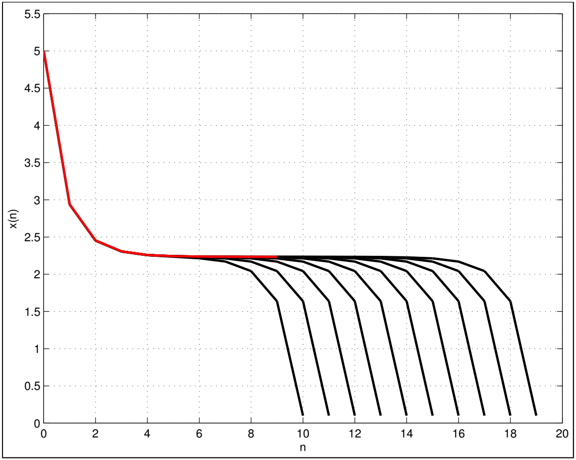

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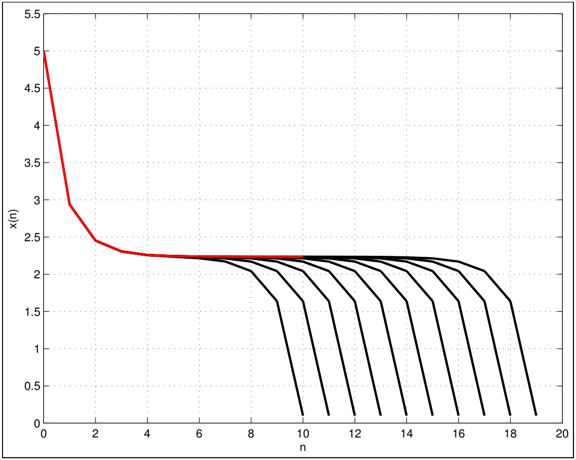

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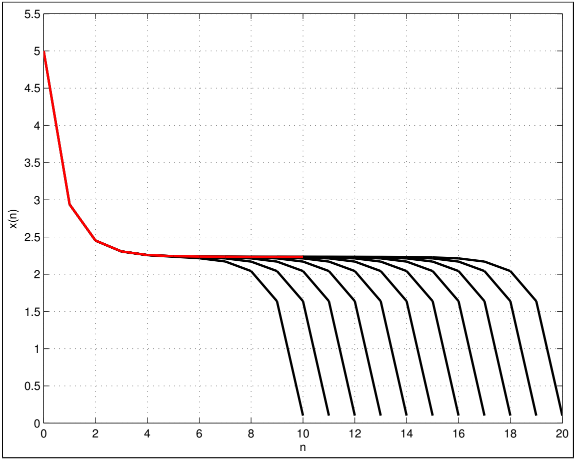

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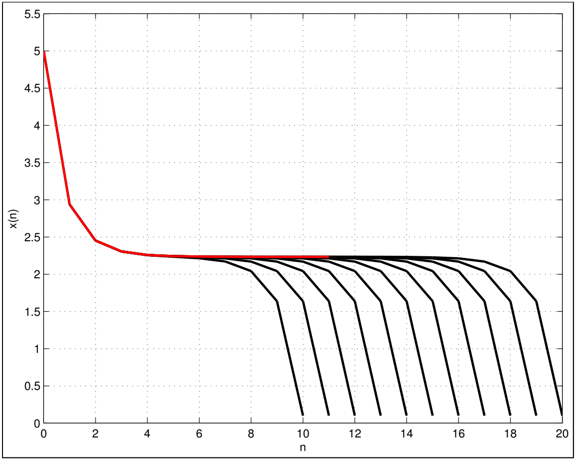

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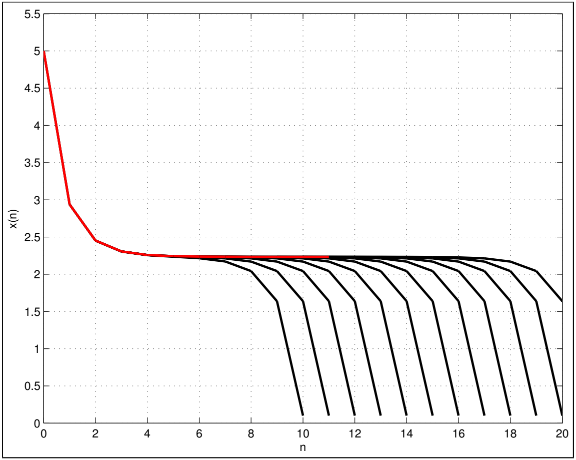

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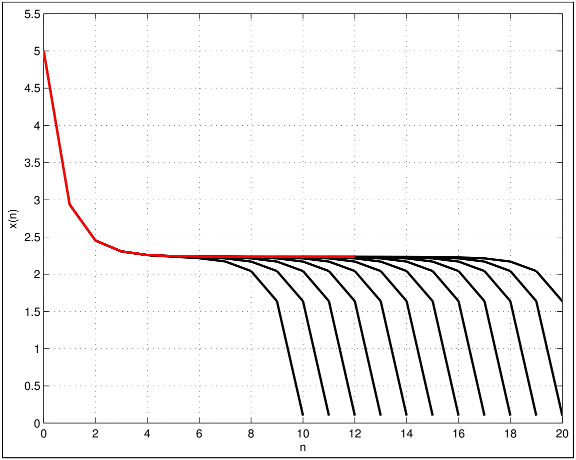

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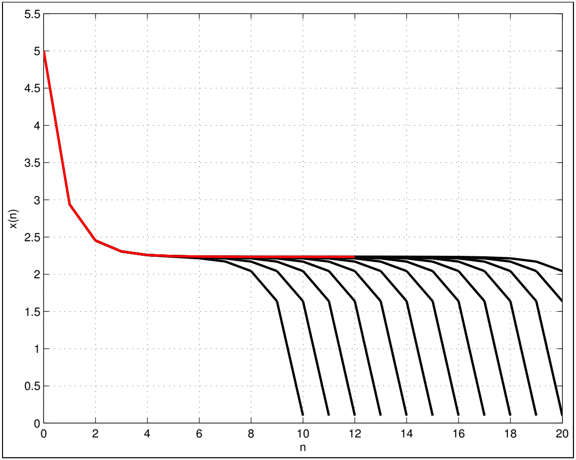

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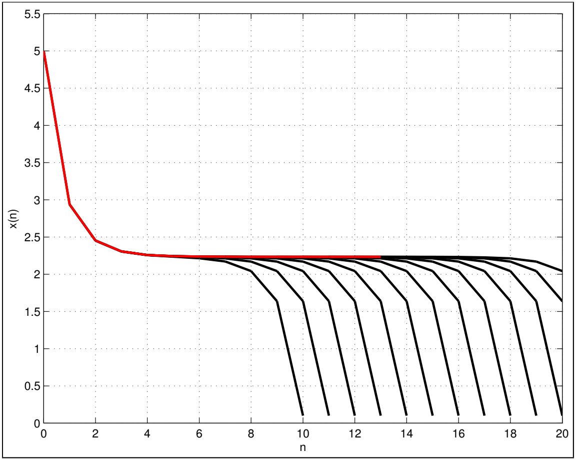

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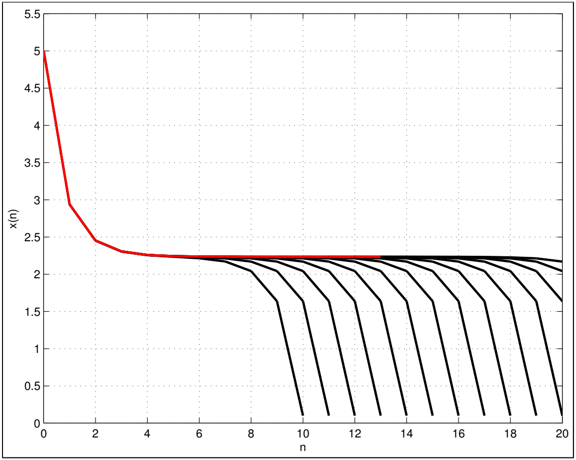

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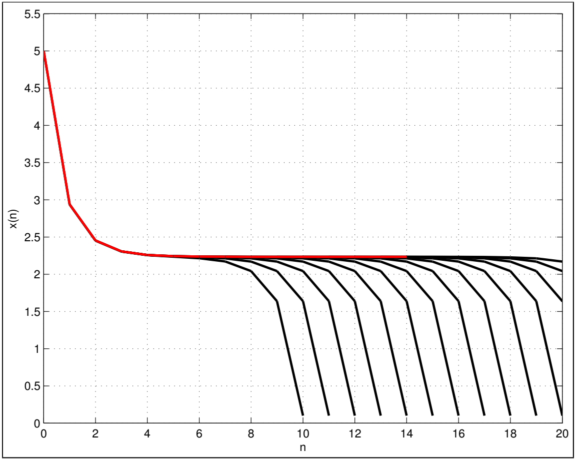

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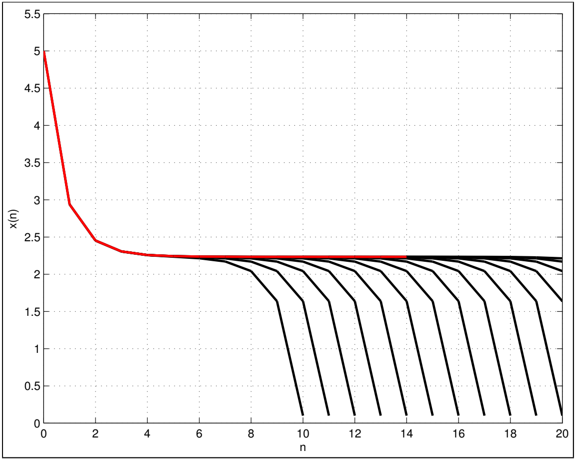

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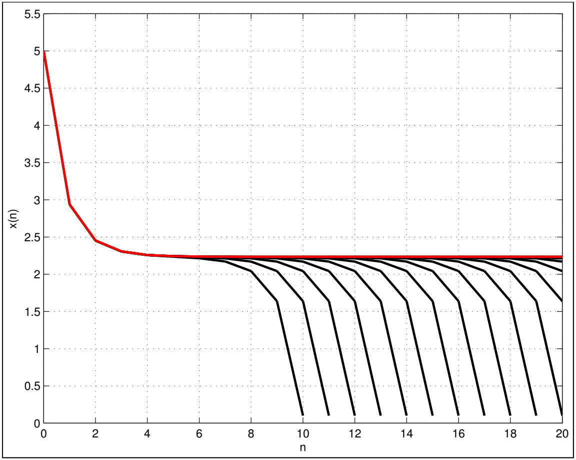

Known results

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We say that x^e is cheaply reachable if there is $E > 0$ such that $V_N(x) \leq N\ell(x^e, u^e) + E$ for all $x \in X$, $N \in \mathbb{N}$

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Theorem [Gr. '13] If the system is strictly dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e, u^e)$ and bounded storage function and x^e is cheaply reachable, then the near optimal turnpike property holds

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(In fact, similar statements can be found in earlier papers and monographs, e.g. in [Carlson/Haurie/Leizarowitz '91])

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Idea of proof: Strict dissipativity implies that the rotated cost

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u))$$

satisfies $\tilde{\ell}(x,u) \geq \alpha(\|x - x^e\|)$.

Known results

We say that x^e is cheaply reachable if there is $E > 0$ such that $V_N(x) \leq N\ell(x^e, u^e) + E$ for all $x \in X$, $N \in \mathbb{N}$

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$\Rightarrow$ if $x_\mathbf{u}$ stays away from x^e for $K \leq N$ steps, $J_N(x,\mathbf{u}) - N\ell(x^e,u^e)$ grows unboundedly as $K \to \infty$, contradicting near optimality

Question

Conclusion: Strict dissipativity can be used as a checkable condition for the turnpike property

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In fact, the theorem just presented relies on another theorem which does not require cheap reachability

Known results

Near equilibrium turnpike property: There is $C > 0$ and $\rho \in \mathcal{K}_\infty$ such that for all $x \in X$, $N \in \mathbb{N}$ and $\delta > 0$, all trajectories $x_\mathbf{u}$ with $J_N(x, \mathbf{u}) \leq N\ell(x^e, u^e) + \delta$ and all $\varepsilon > 0$, the number

$$Q_\varepsilon := \#\{k \in \{0, \dots, N-1\} \mid \|x_\mathbf{u}(k) - x^e\| \leq \varepsilon\}$$

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New results

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Theorem: The following statements are equivalent

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(c) The near equilibrium turnpike property holds and the system is dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function

In other words, the near equilibrium turnpike property exactly closes the gap between dissipativity and strict dissipativity

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Proof idea

We need to prove the equivalences of

(a) strict dissipativity
(b) near equilibrium turnpike and uniform near optim. of x^e
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Question: Can we get rid of dissipativity in (c)?

Yes, if we use that the near equilibrium turnpike property induces an averaged form of optimality of x^e which under additional conditions implies dissipativity

[Müller '14, Müller/Angeli/Allgöwer '13]

New result II

Corollary: Assume X is closed and U is compact, ℓ is continuous and bounded from below, x^e is an equilibrium around which the system is locally controllable and $u^e \in \operatorname{argmin}\{\ell(x^e,u) \,|\, x^e \in X,\ f(x^e,u) = x^e\}$

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Question: Is local controllability really needed?

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Example

$$x^+ = \frac{1}{2}x \qquad \text{and} \qquad \ell(x,u) = u^2 + \frac{\log 2}{\log |x|}$$

The system has (all three kinds of) the turnpike property at $x^e = 0$, because all solutions converge to 0

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However, one computes that the available storage satisfies

$$\lambda(x) := \sup_{K,u} \sum_{k=0}^{K-1} -\Big(\ell(x(k),u(k)) - \ell(x^e,u^e)\Big) = \infty,$$

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Since all other assumptions of the previous corollary are satisfied, it is the lack of controllability which makes its statement fail

Towards new result III

Recall the first two equivalences in the first theorem:

Theorem: The following statements are equivalent

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Yes, but again we need additional assumptions

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Note: the implication "(b) $\Rightarrow$ strict dissipativity" also holds without assuming local controllability

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There is an obvious analogy between the equivalences

turnpike property $\Leftrightarrow$ existence of a storage function

and

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In contrast, the existence of a Lyapunov function yields asymptotic stability for all trajectories for which its defining inequality holds

Application: shape of turnpike trajectories

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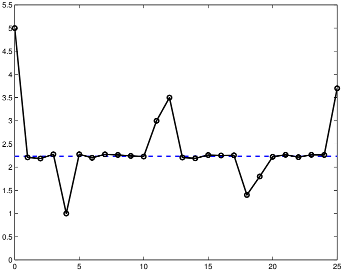
Hence, according to the definition, a turnpike trajectory could look like this



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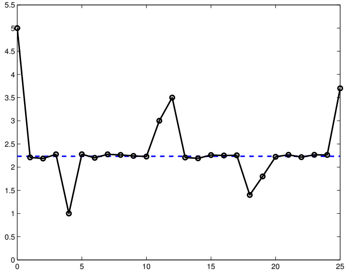
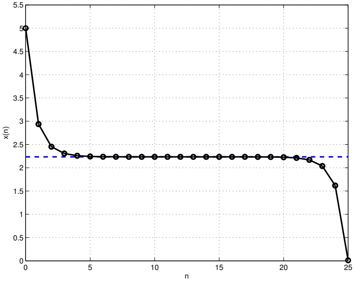


However, in practice in many examples turnpike trajectories look like this



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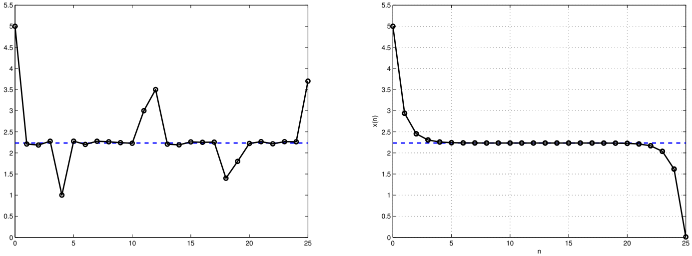

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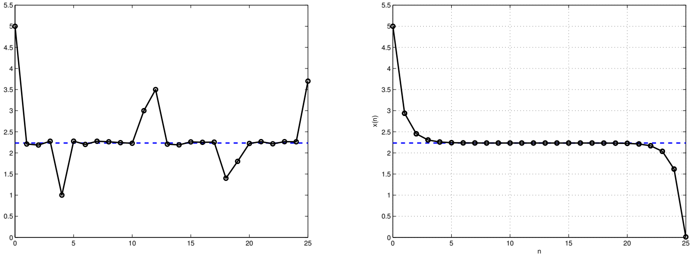

This is because under the stated conditions the turnpike property implies strict dissipativity, which in turn implies stability of the optimal trajectories, in the sense that if $x^{\star}(k) \approx x^e$ then $x^{\star}(k+p) \approx x^e$ for k, p sufficiently small relative to N [Gr. 13]

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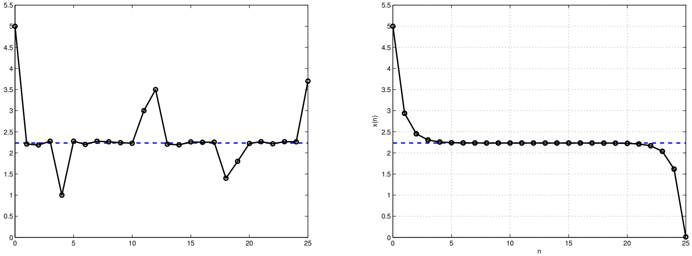

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This excludes excursions from x^e except at the end of the optimal trajecory

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- The results precisely describe the gap between strict dissipativity and turnpike properties
- As a consequence, assuming strict dissipativity for ensuring the turnpike property does not seem overly conservative

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