presented by **Mircea Lazar** in

# FNT2015

## Fukuoka Workshop on Nonlinear Control Theory 2015

**December 13, 2015, Fukuoka, Japan**

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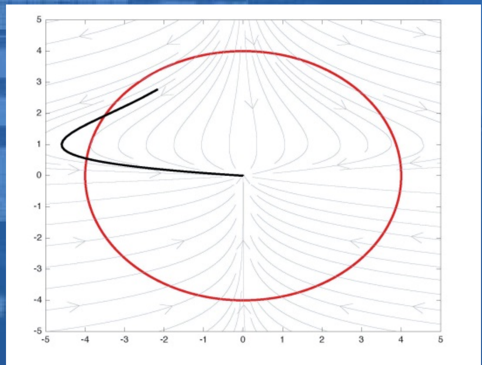
# Workshop on Nonlinear Control Theory

**- 13 December, 2015, Fukuoka, Japan -**

# On asymptotic stability:
# The essence of convergence



*Mircea Lazar, Alina Doban*

Technische Universiteit
**Eindhoven**
University of Technology

**Where innovation starts**

## *Preliminaries: System dynamics*

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary map with $f(0) = 0$. This map can be used to describe a dynamical system:

$$\dot{x}(t) = f(x(t)), \quad \forall t \in \mathbb{R}_+, \quad \forall x(0) \in \mathbb{R}^n$$
$$x(t+1) = f(x(t)), \quad \forall t \in \mathbb{N}, \quad \forall x(0) \in \mathbb{R}^n$$

Let $x(t) = \phi(t, x(0))$ denote the solution at time $t$

If $t \in \mathbb{N}$, $x(t) = f^t(x(0)) := f \circ \ldots \circ f(x(0))$

# *Preliminaries: KL-stability*

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The dynamical system is $\mathcal{KL}$-stable in $\mathcal{S} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{S})$ if:

$$\exists \beta \in \mathcal{KL} \; : \; \|x(t)\| \leq \beta(\|x(0)\|, t), \quad \forall t \in \mathbb{R}_+ (\forall t \in \mathbb{N}), \quad \forall x(0) \in \mathcal{S}$$

If $\mathcal{S} = \mathbb{R}^n$ the property is called global $\mathcal{KL}$-stability

# *Preliminaries: Global Asymptotic Stability*

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If $f$ is continuous $\mathcal{KL}$-stability is equivalent with GAS:

$$\text{Stability: } \forall \varepsilon, \exists \delta(\varepsilon) \; : \; \|x_0\| \leq \delta(\varepsilon) \Rightarrow \|x(t)\| \leq \varepsilon, \quad \forall t$$

$$\text{Convergence: } \lim_{t \to \infty} \|x(t)\| = 0 \text{ for all } x(0) \in \mathcal{S} \subseteq \mathbb{R}^n$$

# *Preliminaries: Lyapunov functions*

A real-valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a **Lyapunov function** if:

**P1.** $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ :

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n$$

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**b2.** $\exists \alpha_3 \in \mathcal{K} : V(x(t+1)) \leq V(x(t)) - \alpha_3(\|x(t)\|), \quad \forall x(0) \in \mathcal{S}$

**Properties P2. b1. and P2. b2. are equivalent:**

- If P2. b2. holds, P2. b1. holds with $\rho(s) \leq \tilde{\rho}(s) := (\text{id} - 0.5\alpha_3 \circ \alpha_2^{-1})(s) < \text{id}(s)$
- If P2. b1. holds, P2. b2. holds with $\tilde{\alpha}_3(s) := (\text{id} - \rho) \circ \alpha_1(s) \leq \alpha_3(s) \in \mathcal{K}$

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**Remarks:**

- Existence of a Lyapunov function $V$ implies $\mathcal{KL}$-stability in $\mathcal{S}$, under the assumption that $\mathcal{S}$ is an invariant set
- $\mathcal{KL}$-stability in $\mathcal{S}$ implies existence of a Lyapunov function $V$, but for a specific dynamics $f$, it is not known which type of $V$ is non-conservative

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**Remarks:**

- Convergence condition is now unified; distinction comes from the nature of time and solution
- This relaxation was originally proposed by:

D. Aeyels and J. Peuteman, A new asymptotic stability criterion for nonlinear time-variant differential equations, IEEE Transactions on Automatic Control, vol. 43, no. 7, pp. 968-971, 1998

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**Remarks:**

- Property P2. a. implies $\dot{V}(x(t)) < 0$ when $d \to 0$
- Property P2. b. also recovers the standard decrease condition when $d = 1$

## *Instrumental result: K-infinity bounds on positive functions*

Let $W : \mathbb{R}^n \to \mathbb{R}_+$ with $W(0) = 0$ and $W(x) \to \infty$ as $x \to \infty$ be a positive definite and continuous function on $\mathbb{R}^n$.

Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

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**Remarks:**

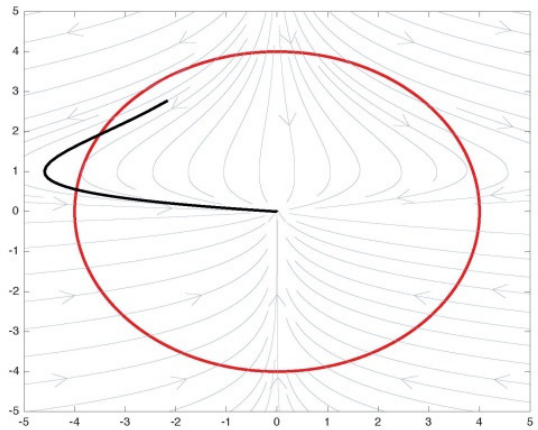
- Originally formulated by W. Hahn, *Stability of motion*, 1967
- The proof boils down to upper and lower bounding positive definite, continuous and non-decreasing functions by class $\mathcal{K}_\infty$ functions
- A possible explicit construction of the lower bound is worked out in:

M. Lazar, W.P.M.H. Heemels, A.R. Teel, Further input-to-state stability subtleties for discrete-time systems, IEEE Transactions on Automatic Control, vol. 58, no. 6, pp. 1609-1613, 2013

# Main results: FTLFs imply KL-stability

Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a FTLF in some set $\mathcal{S} \subseteq \mathbb{R}^n$. Assume that $\mathcal{S}$ is $d$-invariant for the dynamics $f$.

Then the dynamics are $\mathcal{KL}$-stable in $\mathcal{S}$ with respect to the zero equilibrium.



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*Proof sketch:* For any $t \in \mathbb{R}_+$ there exists $N \in \mathbb{N}$ and $j \in \mathbb{R}_+$, $j < d$ such that $t = Nd + j$.

$$\begin{aligned} V(x(t)) &= V(x(Nd+j)) = V(x(((N-1)d+j)+d)) \\ &\leq \rho(V(x((N-1)d+j))) \\ &\dots \\ &\leq \rho^N(V(x(j))) \leq \rho^N(\alpha_2(\|x(j)\|)) \end{aligned}$$

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*Proof sketch:* Solution is given by:

$$x(j) = x(0) + \int_0^j f(x(s))ds, \quad \forall j \in \mathbb{R}_+$$

Define $\max_{j \in [0,d]} \|x(j)\| =: F_d(x(0))$.

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Hence, by continuity of solutions on the initial condition:

$$\forall j \in \mathbb{R}_{[0,d]} \ : \ \|x(j)\| \leq \|F_d(x(0))\| \leq \omega(\|x(0)\|), \quad \omega \in \mathcal{K}_\infty$$

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$$
\begin{aligned}
\boxed{V(x(t)) \le \rho^N(\alpha_2(\|x(j)\|))} &\le \rho^N(\hat{\alpha}_2(\|x(0)\|)) \\
&\le \rho^{\lfloor \frac{t}{d} \rfloor - 1} \circ \rho^{-1} \circ \hat{\alpha}_2(\|x(0)\|) \\
&\le \rho^{\lfloor \frac{t}{d} \rfloor} \circ \hat{\rho} \circ \hat{\alpha}_2(\|x(0)\|), \quad \hat{\rho} \in \mathcal{K}_\infty \\
&=: \hat{\beta}(\|x(0)\|, t)
\end{aligned}
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\end{aligned}$$

Finally, this yields by inverting the lower bound on $V$:

$$\|x(t)\| \le \alpha_1^{-1} \circ \hat{\beta}(\|x(0)\|, t) =: \beta(\|x(0)\|, t) \in \mathcal{KL}$$

# *Main results: FTLF converse theorem*

Let the dynamics $f$ be $\mathcal{KL}$-stable with respect to the origin and some proper invariant set $\mathcal{S} \subseteq \mathbb{R}^n$.

Suppose that there exists a $d \in \mathbb{R}_{>0}$ such that $\boxed{\beta(s,d) < s}$ for all $s > 0$.

Let $V(x) = \eta(\|x\|)$, where $\eta \in \mathcal{K}_\infty$ can be taken arbitrarily.

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*Proof sketch:*
$$\begin{aligned} V(x(t+d)) &= \eta(\|x(t+d)\|) \leq \eta(\beta(\|x(t)\|, d)) \\ &\leq \eta(\beta(\eta^{-1}(V(x(t))), d)) \\ &=: \rho(V(x(t))) \end{aligned}$$

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By the assumption on the $\mathcal{KL}$-function $\beta$ it holds that:

$$\rho(s) = \eta(\beta(\eta^{-1}(s), d)) < \eta(\eta^{-1}(s)) = \mathrm{id}(s)$$

## *Main results: Alternative Lyapunov converse theorem*

Define $W(x(t)) := \int_t^{t+d} V(x(\tau))d\tau$. Define $W(x(t)) = \sum_{j=t}^{t+d-1} V(x(j))$.

$V$ is a $d$-finite time Lyapunov function if and only if $W$ is a Lyapunov function (for the dynamics $f$ with respect to the zero equilibrium).

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*Proof sketch:* By the Leibniz integral rule we have that:

$$\begin{aligned}\frac{d}{dt}W(x(t)) &= \int_t^{t+d} \frac{d}{dt}V(x(\tau))d\tau + V(x(t+d))(\dot{t+d}) - V(x(t))\dot{t} \\ &= V(x(t+d)) - V(x(t)) \\ &\leq -\rho(\|x(t)\|)\end{aligned}$$

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**Remark:**

- The discrete-time result was proven first in:

R. Geiselhart, R.H. Gielen, M. Lazar, F.R. Wirth, An Alternative Converse Lyapunov Theorem for Discrete-Time Systems, Systems & Control Letters, 70, 49-59, 2014.

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**Remarks:**

- Original Massera construction: $W(x(t)) = \int_t^{\infty} \alpha(\|x(\tau)\|) d\tau$
- Under GES assumption, H. Khalil proposed: $W(x(t)) = \int_t^{t+N} \|x(\tau)\|_2 d\tau$

# *Instrumental result: Expansion of LFs and FTLFs*

Let $W$ be a LF. Define $W_1(x) = W(x + \alpha_1 f(x))$.

Define $\mathcal{S}_W(c) := \{x \in \mathbb{R}^n \ : \ W(x) \leq c\}$ and similarly $\mathcal{S}_{W_1}(c)$.

Then $W_1$ is a LF and $\mathcal{S}_W(c) \subset \mathcal{S}_{W_1}(c)$.

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Let $V$ be a $d$-FTLF. Define $V_1(x) = V(x + \alpha_1 f(x))$.

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# *Constructive methodology: Main ideas*

The developed results can be used to construct LFs and DOAs as follows:

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**Remarks:**

- At step 1. one can work with a linearization of $f$
- In this case taking $\eta = \mathrm{id}$ yields: $\|e^{d\left[\frac{\partial f(x)}{\partial x}\right]_{x=0}}\| < 1$

# *Illustrative example 1:* Bilinear dynamics – no polynomial LF

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{pmatrix}$$

A. Ahmadi, M. Krstic, P.Parrilo, A globally asymptotically stable vector field with no polynomial Lyapunov function, IEEE CDC, 2011, pp. 7579-7580.

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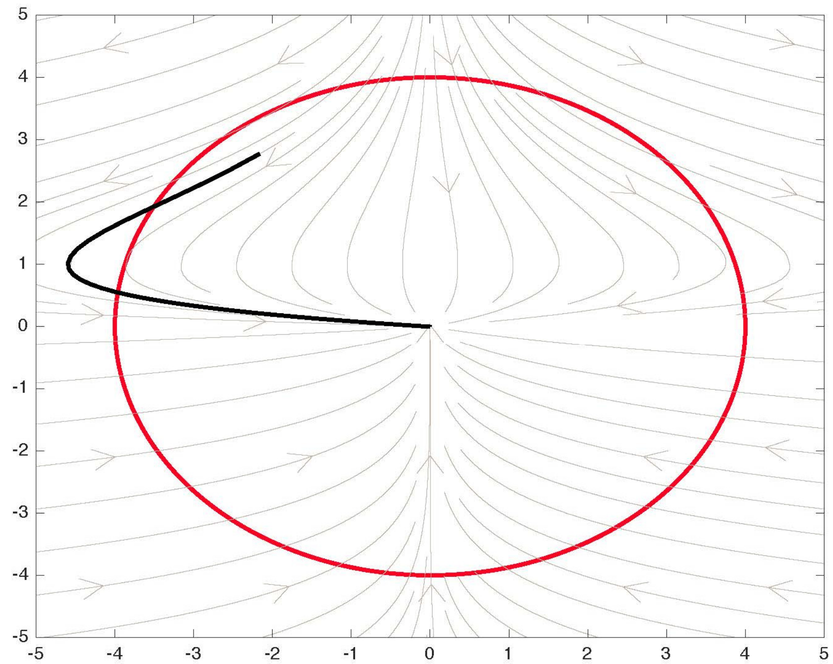


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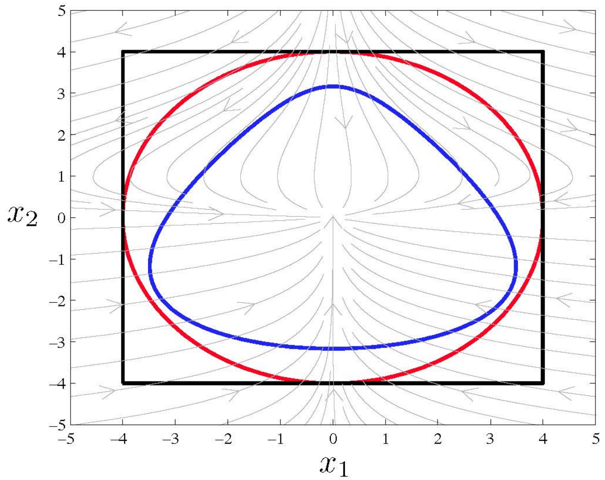


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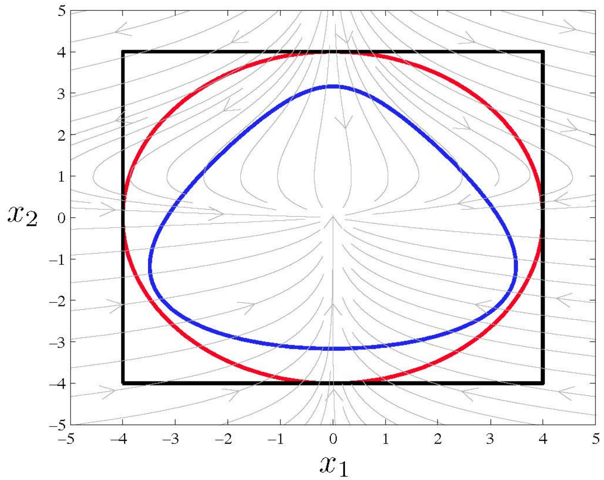


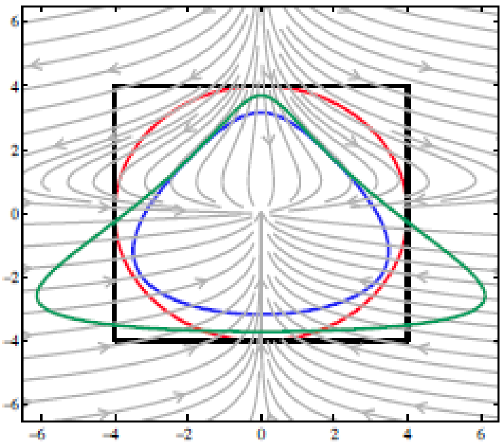

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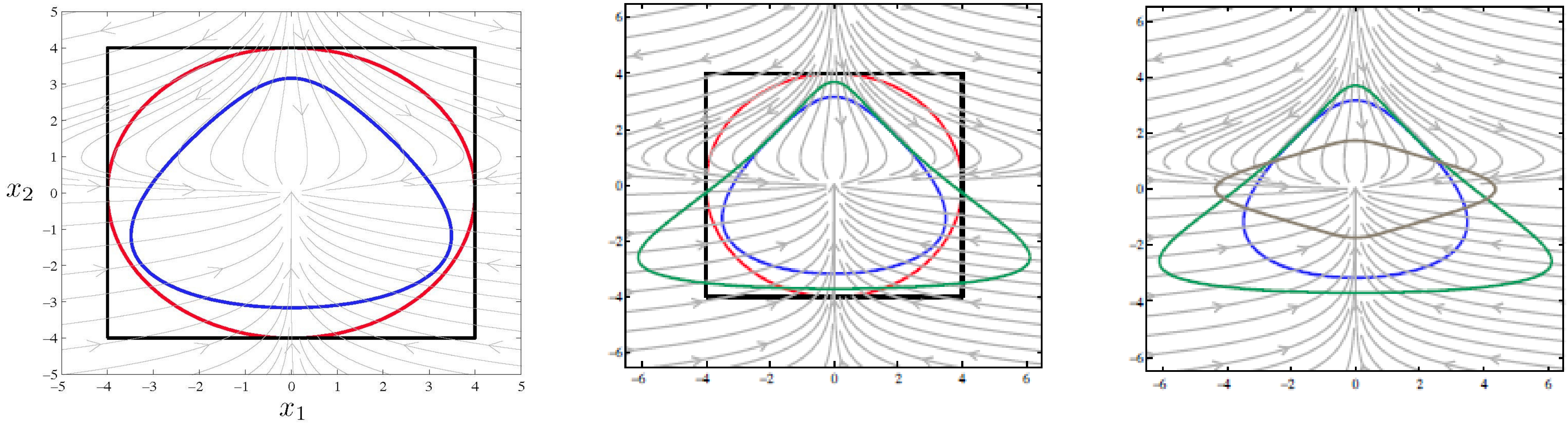


$$V_{GAS}(x) = \ln(1 + x_1^2) + x_2^2$$

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## *Illustrative example 2: 3D example from literature*

$$\dot{x} = f(x) = \begin{pmatrix} x_1(x_1^2 + x_2^2 - 1) - x_2(x_3^2 + 1) \\ x_2(x_1^2 + x_2^2 - 1) + x_1(x_3^2 + 1) \\ 10x_3(x_3^2 - 1) \end{pmatrix}$$

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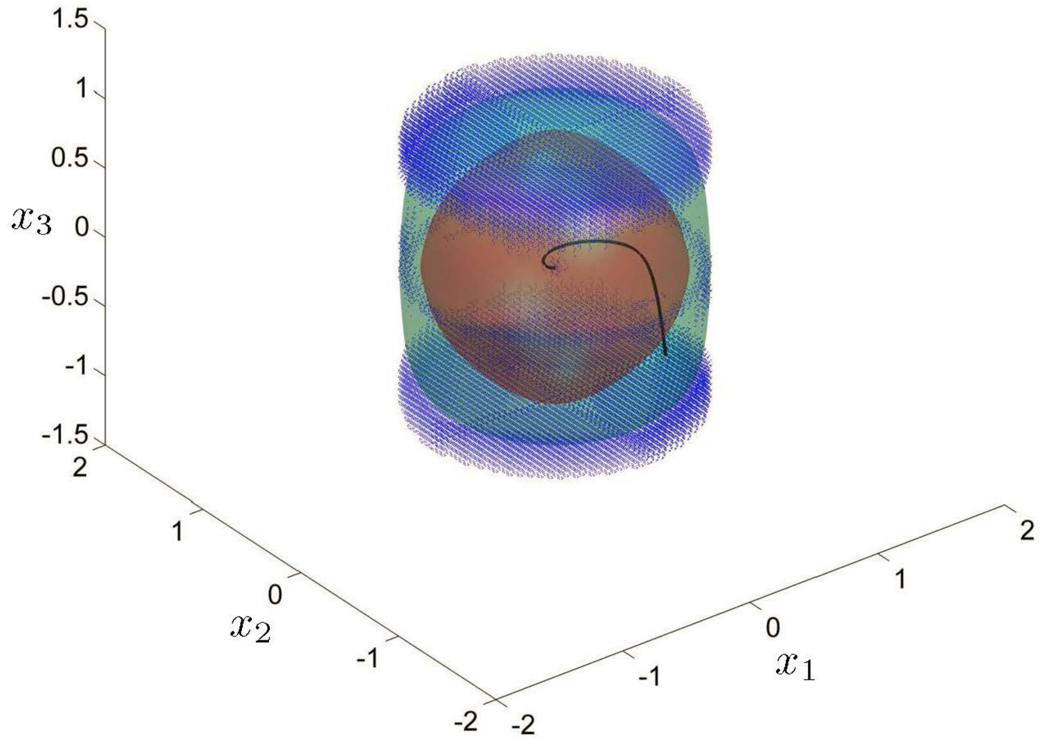


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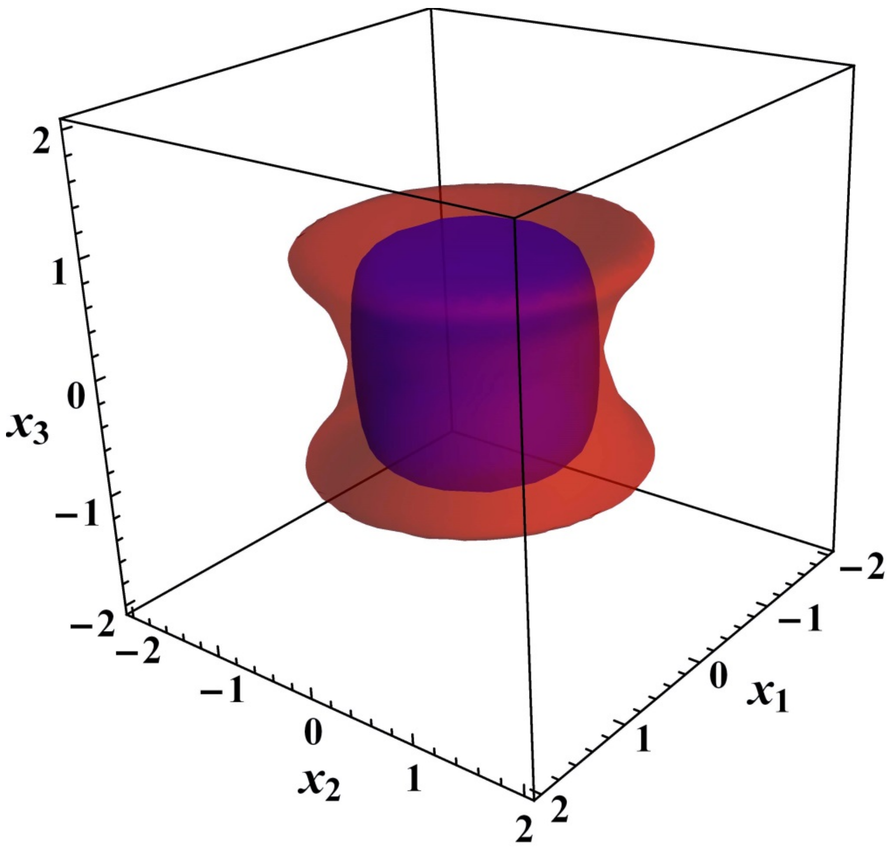


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# *Illustrative example 3: Nonpolynomial 2D – genetic toggle switch*

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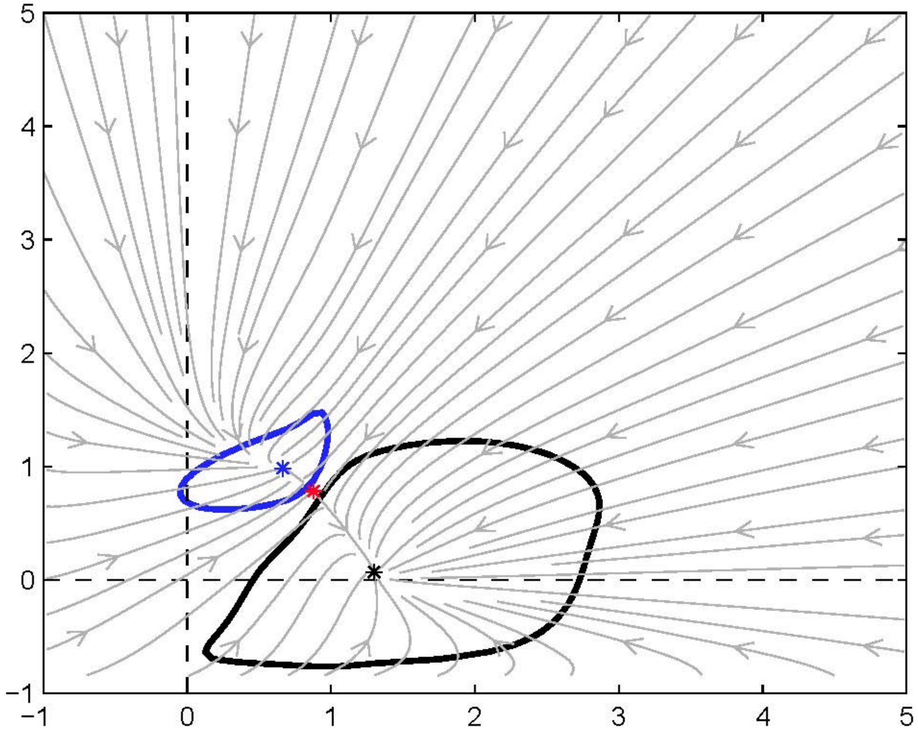


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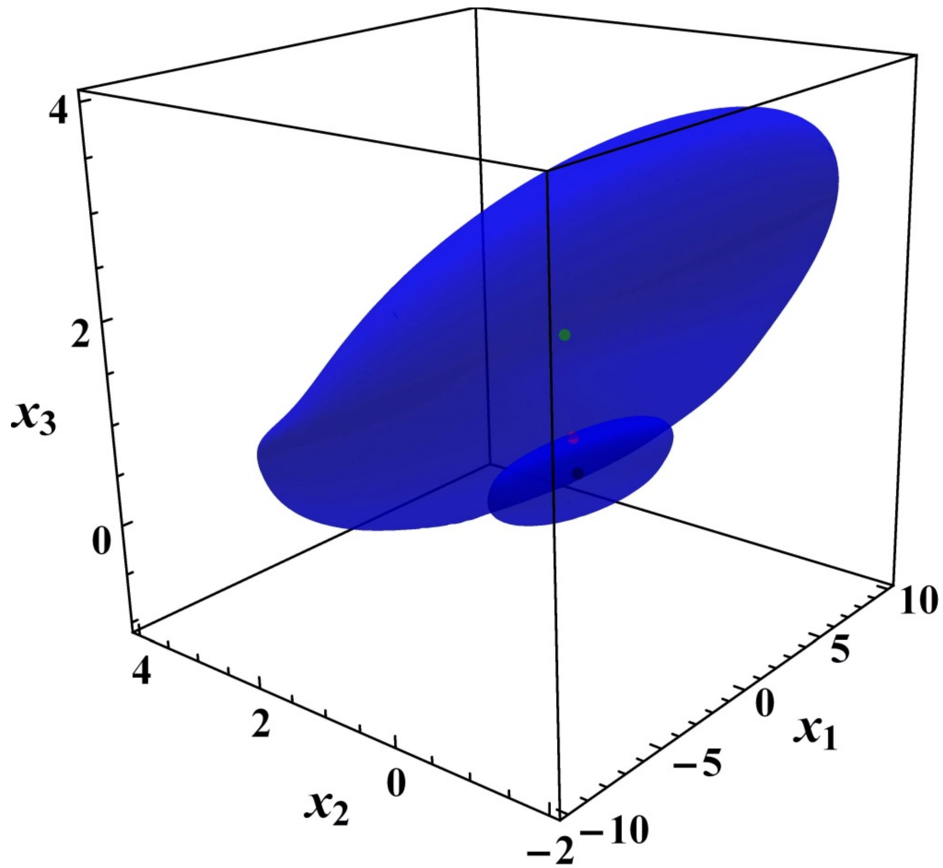
# *Illustrative example 4: Nonpolynomial 3D – HPA axis*

$$\dot{x} = f(x) = \begin{pmatrix} \left(1 + \xi \frac{x_3^\alpha}{1+x_3^\alpha} - \psi \frac{x_3^\gamma}{x_3^\gamma + \tilde{c}_3^\gamma}\right) - \tilde{\omega}_1 x_1 \\ \left(1 - \rho \frac{x_3^\alpha}{1+x_3^\alpha}\right) - \tilde{\omega}_2 x_2 \\ x_2 - \tilde{\omega}_3 x_3 \end{pmatrix}$$

M. Andersen, F. Vinther, J.T. Ottesen, Mathematical modeling of the hypothalamicpituitaryadrenal gland (hpa) axis, including hippocampal mechanisms, Mathematical Biosciences, 2013.

# *Illustrative example 4: Nonpolynomial 3D – HPA axis*

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# *Concluding remarks*

**Summary of relevant features:**

- There is more freedom in choosing the candidate FTLF
- Analytical formula for $W$ improves scalability
- Knowledge of solution $x(d)$ is tackled by linearization
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**References with technical details:**

- Discrete-time case (due to Roman, during a visit in our group at TU/e):

  R. Geiselhart, R.H. Gielen, M. Lazar, F.R. Wirth, An Alternative Converse Lyapunov Theorem for Discrete-Time Systems, Systems & Control Letters, 70, 49-59, 2014.

- Continuous-time case results and examples (due to Alina):

  A.I. Doban, M. Lazar, Computation of Lyapunov functions for nonlinear differential equations via a Massera-type construction, submitted to ECC 2016.


*Thank you for your attention!*
*Special thanks to Hiroshi for all the efforts*