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Sampled-Data Control of Nonlinear Retarded Systems

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Stabilization in the Sample-and-Hold Sense


Systems Described by RFDEs

\[ \dot{x}(t) = f(x_t, u(t)), \quad t \geq 0, \ a.e., \]
\[ x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C, \]  

(1)

\( x(t) \in R^n, \ n \) is a positive integer; \( \Delta \) is a positive integer, the maximum involved time-delay; \( C \) is the Banach space of continuous functions mapping \([−\Delta, 0] \) to \( R^n \), endowed with the norm of uniform topology, denoted with \( \| \cdot \|_\infty \); \( x_t \in C \) is defined as \( x_t(\tau) = x(t + \tau), \ \tau \in [-\Delta, 0] \); \( f \) is a map from \( C \times R^m \) to \( R^n \), Lipschitz on bounded sets, zero at zero; \( m \) is a positive integer; \( u(t) \in R^m \) is a Lebesgue measurable, locally essentially bounded signal.

For a positive real \( r \), \( C_r = \{ \phi \in C : \|\phi\|_\infty \leq r \} \)
Definition 1. Let $V : \mathcal{C} \to R^+$ be a locally Lipschitz functional. The derivative $D^+V : \mathcal{C} \times R^m \to R^*$ of the functional $V$ is defined, in the Driver’s form (see Driver, 1962, Burton, 1985, P. & Jiang, 2006, Karafyllis, 2006), for $\phi \in \mathcal{C}$, $v \in R^m$, as follows

$$D^+V(\phi, v) = \lim_{h \to 0^+} \frac{1}{h} \left( V(\phi_{h,v}) - V(\phi) \right),$$

where $\phi_{h,v} \in \mathcal{C}$ is given by

$$\phi_{h,v}(s) = \begin{cases} 
\phi(s + h), & s \in [-\Delta, -h], \\
\phi(0) + f(\phi, v)(h + s), & s \in (-h, 0]
\end{cases}.$$
Theorem 2. (Karafyllis, P., Jiang, EJC 2008) Let in the RFDE (1) \( u(t) = 0, \ t \geq 0 \). The system described by the RFDE (1) is 0–GAS if and only if there exist a locally Lipschitz functional \( V : C \rightarrow R^+ \) and functions \( \alpha_1, \alpha_2 \) of class \( \mathcal{K}_\infty \), \( \alpha_3 \) of class \( \mathcal{K} \), such that, \( \forall \phi \in C \), the following inequalities hold:

i) \( \alpha_1(\|\phi\|_\infty) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty) \);

ii) \( D^+ V(\phi, 0) \leq -a_3(\|\phi\|_\infty) \)
Definition 3. (P., SICON 2014) A functional $V : \mathcal{C} \to R^+$ is said to be smoothly-separable if there exist a function $V_1 \in C^1_L(R^n; R^+)$, a locally Lipschitz functional $V_2 : \mathcal{C} \to R^+$, functions $\beta_i$ of class $\mathcal{K}_\infty$, $i = 1, 2$, such that, for any $\phi \in \mathcal{C}$, the following equality/inequality inequalities hold:

$$V(\phi) = V_1(\phi(0)) + V_2(\phi), \quad \beta_1(|\phi(0)|) \leq V_1(\phi(0)) \leq \beta_2(|\phi(0)|)$$
Definition 4. (Artstein, NA 1983, Jankovic, TAC 2001, P., SICON 2014) A smoothly-separable functional $V : \mathcal{C} \rightarrow \mathbb{R}^+$ is said to be a CLKF if there exist functions $\gamma_1, \gamma_2$ of class $\mathcal{K}_\infty$ such that the following inequalities hold

\begin{enumerate}[i)]
  \item $\gamma_1(|\phi(0)|) \leq V(\phi) \leq \gamma_2(\|\phi\|_\infty)$, $\forall \phi \in \mathcal{C}$;
  \item $\inf_{u \in \mathbb{R}^m} D^+V(\phi, u) < 0$, $\forall \phi \in \mathcal{C}$, $\phi(0) \neq 0$.
\end{enumerate}
Definition 5. (P., SICON 2014) A map $k : \mathcal{C} \to U$ (continuous or not) is said to be a steepest descent feedback, induced by a CLKF $V$, if the following condition holds: there exist $m \in \{0, 1\}$, positive reals $\eta$ and $\mu$, a function $p \in C^1_L(R^+; R^+)$, of class $\mathcal{K}_\infty$, such that, $\forall \phi \in \mathcal{C}$,

$$mD^+V(\phi, k(\phi)) + \eta \max\{0, D^+p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \leq 0$$

Recall: $V(\phi) = V_1(\phi(0)) + V_2(\phi)$
\[ \dot{x}(t) = x(t - \Delta) + |x(t)|u(t) \]

\[ V(\phi) = V_1(\phi(0)) + V_2(\phi), \phi \in \mathcal{C} \]
\[ V_1(x) = x^2, \ x \in \mathbb{R}, \quad V_2(\phi) = \int_{-\Delta}^{0} 2\phi^2(\tau)d\tau, \ \phi \in \mathcal{C} \]
\[ k(\phi) = -2sgn(\phi(0)) \]

\( V \) is CLKF, \( k \) is a (discontinuous) steepest descent feedback. Indeed, for \( m = 1, \ \eta = 0.1, \ p = I_d, \ \mu = 1 \), we have, for any \( \phi \in \mathcal{C} \):

\[
\inf_{u \in \mathbb{R}} D^+ V(\phi, u) \leq D^+ V(\phi, k(\phi)) \leq -\phi^2(0) - \phi^2(-\Delta),
\]

\[
mD^+ V(\phi, k(\phi)) + \eta \max\{0, D^+ p \circ V_1(\phi, k(\phi)) + \mu V_1(\phi(0))\} \leq -\phi^2(0) - \phi^2(-\Delta) + 0.1 \max\{0, -2\phi^2(0) + \phi^2(-\Delta)\} \leq 0
\]
Assumption 6. There exists a positive real \( q \) such that the initial condition \( x_0 \in W^{1,\infty} \), and \( \text{ess sup}_{\theta \in [-\Delta_0]} \left| \frac{dx_0(\theta)}{d\theta} \right| \leq q \). There exist a CLKF \( V \) and an induced steepest descent feedback \( k \) (continuous or not). The map \( \phi \to D^+V_2(\phi, u) \) is Lipschitz on bounded subsets of \( C \times R^m \).
Definition 7. (Clarke et al., TAC 1997, P., SICON 2014) A partition \( \pi = \{t_i, \ i = 0,1,\ldots\} \) of \([0, +\infty)\) is a countable, strictly increasing sequence \( t_i \), with \( t_0 = 0 \), such that \( t_i \to +\infty \) as \( i \to +\infty \). The diameter of \( \pi \), denoted \( \text{diam}(\pi) \), is defined as \( \sup_{i \geq 0} t_{i+1} - t_i \). The dwell-time of \( \pi \), denoted \( \text{dwell}(\pi) \), is defined as \( \inf_{i \geq 0} t_{i+1} - t_i \). For any positive reals \( a \in (0,1], \ b > 0 \), \( \pi_{a,b} \) is any partition \( \pi \) with \( ab \leq \text{dwell}(\pi) \leq \text{diam}(\pi) \leq b \).
Definition 8. (Clarke et al., TAC 1997, P., SICON 2014) We say that a feedback $F : \mathcal{C} \rightarrow \mathbb{R}^m$ (continuous or not) stabilizes the system described by the RFDE in the sample-and-hold sense if, for every positive reals $r, R$, $0 < r < R$, $a \in (0, 1]$, there exist a positive real $\delta$ depending upon $r, R, q$ and $\Delta$, a positive real $T$, depending upon $r, R, q, \Delta$ and $a$, and a positive real $E$, depending upon $R$ and $\Delta$, such that, for any partition $\pi_{a,\delta} = \{t_i, \, i = 0, 1, \ldots\}$, for any initial state $x_0 \in \mathcal{C}_R$, the solution corresponding to $x_0$ and to the sampled-data feedback control law $u(t) = F(x_{t_k}), \, t_k \leq t < t_{(k+1)}, \, k = 0, 1, \ldots$, exists $\forall t \geq 0$ and, furthermore, satisfies:

$$x_t \in \mathcal{C}_E, \, \forall t \geq 0; \quad x_t \in \mathcal{C}_r, \, \forall t \geq T$$
Theorem 9. (P., SICON 2014) Any steepest descent feedback $k$ (continuous or not) stabilizes the system described by (1) in the sample-and-hold sense.
An Example from Sliding Mode Control

delay-free case studied in Khalil’s book.

\[ \begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= H(x_t) + G(x_t)u(t), \\
x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0],
\end{align*} \]

where: \( x(t) = [x_1(t) \ x_2(t)]^T \in R^2; \) \( \Delta \) is an arbitrary positive real; \( H : C \rightarrow R, \ G : C \rightarrow R^+ \) are uncertain maps, Lipschitz on bounded sets; \( H(0) = 0; \) \( x_0 \in W^{1,\infty}, \ ess \sup_{\theta \in [-\Delta, 0]} \left| \frac{dx_0(\theta)}{d\theta} \right| \leq q; \) \( q \) is an arbitrary positive constant; \( u(t) \in R \) is the control input.
We introduce the following standard assumption.

1) there exists a positive real $g_0$ such that, for all $\phi \in \mathcal{C}$, the inequality holds $G(\phi) \geq g_0$;

2) there exist a positive real $a_1$, a locally bounded function $\rho : \mathcal{C} \to \mathbb{R}^+$ such that, for all $\phi \in \mathcal{C}$, the inequality holds $|a_1\phi_2(0) + H(\phi)| \leq \rho(\phi)G(\phi)$
Let us consider the Lyapunov-Krasovskii functional $V : C \to \mathbb{R}^+$ defined, for $\phi = [ \phi_1 \phi_2 ]^T \in C$, as

$$V(\phi) = (a_1 \phi_1(0) + \phi_2(0))^2 + \begin{cases} 
\frac{1}{2} \gamma \phi_1^2(0), & |\phi_1(0)| \leq 1, \\
\gamma \left( |\phi_1(0)| - \frac{1}{2} \right), & |\phi_1(0)| > 1,
\end{cases}$$

with $\gamma$ a suitable positive parameter which will be chosen later. Such functional is a CLKF. Indeed, for any $\phi = [ \phi_1 \phi_2 ]^T \in C$, $u \in \mathbb{R}$, by the equality

$$\phi_1(0)\phi_2(0) = \phi_1(0)(a_1 \phi_1(0) + \phi_2(0)) - a_1 \phi_1^2(0),$$

we have

$$D^+ V(\phi, u) \leq 2G(\phi) (a_1 \phi_1(0) + \phi_2(0)) \left( \frac{a_1 \phi_2(0) + H(\phi)}{G(\phi)} + u \right) + \gamma |a_1 \phi_1(0) + \phi_2(0)| - \gamma a_1 \min \left\{ |\phi_1(0)|, \phi_1^2(0) \right\}$$
Taking into account of the possibility of choosing (a sliding mode control feedback) \( u = -(\rho(\phi) + k_0) \cdot sgn(a_1\phi_1(0) + \phi_2(0)) \), with \( k_0 \) a positive real, we have, for all \( \phi \in C \),

\[
\inf_{u \in \mathbb{R}^D} D^+ V(\phi, u) \leq D^+ V(\phi, -(\rho(\phi) + k_0) \cdot sgn(a_1\phi_1(0) + \phi_2(0))) \leq -(2g_0k_0 - \gamma)|a_1\phi_1(0) + \phi_2(0)| - \gamma a_1 \min \{|\phi_1(0)|, \phi_1^2(0)\}.
\]

Therefore, by choosing any \( \gamma \in (0, 2g_0k_0) \), it follows that \( \inf_{u \in \mathbb{R}} D^+ V(\phi, u) < 0, \forall \phi \in C, \phi(0) \neq 0 \). Let the map \( k : C \to \mathbb{R} \) be defined, for \( \phi \in C \), as

\[
k(\phi) = -(\rho(\phi) + k_0) sgn(a_1\phi_1(0) + \phi_2(0))
\]
The map $k$ is a steepest descent feedback. Indeed, let
$m = \mu = 1$, $s \geq 0$, $\eta = \min \{2g_0k_0 - \gamma, \ a_1\}$, $p(s) = \log n(1 + s)$,
$s \geq 0$. We have, for any $\phi \in C$, taking into account of the
increasing property of the function $p$, and that $V(\phi) = V_1(\phi(0))$,

$$D^+ V(\phi, k(\phi)) + \eta \max \{0, \ D^+ p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \leq$$
$$- \min \{2g_0k_0 - \gamma, \ a_1\} \left( |a_1\phi_1(0) + \phi_2(0)| + \gamma \min \{|\phi_1(0)|, \ \phi_1^2(0)\} \right)$$
$$+ \min \{2g_0k_0 - \gamma, \ a_1\} \log n \left( 1 + (a_1\phi_1(0) + \phi_2(0))^2 \right)$$
$$+ \gamma \min \{|\phi_1(0)|, \ \phi_1^2(0)\} \right)$$

By the inequality $\log n(1 + s_1^2 + s_2) - s_1 - s_2 \leq 0$, $\forall \ s_1, s_2 \in R^+$, it
follows that, $\forall \phi \in C$,

$$D^+ V(\phi, k(\phi)) + \eta \max \{0, \ D^+ p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \leq 0,$$
that is, $k$ is a steepest descent feedback.
We conclude that the steepest descent feedback $k$ stabilizes the system in the sample-and-hold sense. The piece-wise constant control law is defined as follows, for $t \geq 0$,

$$u(t) = -\left(\rho \left(x_{t_k}\right) + k_0\right) \text{sgn}(a_1 x_1(t_k) + x_2(t_k)),$$

$$t_k \leq t < t_{(k+1)}, \quad k = 0, 1, \ldots, \quad t_0 = 0.$$
Simulations have been performed with $H$, $G$ defined, for 
\[ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C, \] as 
\[ H(\phi) = b_1 \phi_1(-\Delta)\phi_2(-\Delta), \quad G(\phi) = b_2, \] 
where $b_i$, $i = 1, 2$ are uncertain parameters, $b_1 \in [-1, 1]$, 
$b_2 \in [1, 2]$, $\Delta$ is a known positive constant. We can choose, 
in this case, $a_1 = 1$, $\rho(\phi) = |\phi_1(-\Delta)\phi_2(-\Delta)| + |\phi_2(0)|$, 
$\phi \in C$. In the performed simulations, $k_0 = 0.1$, $\Delta = 1.4$, 
\[ x_0(\tau) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \tau \in [-\Delta, 0], \quad a = 1 \quad \text{(uniform sampling)}, \quad b_1 = -1, \] 
$b_2 = 1$ are chosen. In simulations, a disturbance $d(t) = d_k$, 
k$\delta \leq t < (k + 1)\delta$, $k = 0, 1, \ldots$, adding to the control law, is also 
considered. Such disturbance is generated at each sampling 
time as an element of the interval $[-0.15, 0.05]$ with uniform 
probability density function.
Figure 1: Variables $x_1$ and $x_2$, $\delta = 0.3$
Figure 2: Input Signal (plus disturbance), $\delta = 0.3$
Local Results

Definition 10. (P., SICON 2014) Let $Q$ be a positive real. We say that a feedback $F : \mathcal{C}_Q \to \mathbb{R}^m$ (continuous or not) stabilizes the system described by the RFDE in the sample-and-hold sense, in $\mathcal{C}_Q$, if, for every positive reals $r$, $R$, $0 < r < R \leq Q$, $a \in (0, 1]$, there exist a positive real $\delta$ depending upon $r$, $R$, $q$ and $\Delta$, a positive real $T$, depending upon $r$, $R$, $q$, $\Delta$ and $a$, and a positive real $E$, depending upon $R$ and $\Delta$, such that, for any partition $\pi_{a,\delta} = \{t_i, \ i = 0, 1, \ldots \}$, for any initial state $x_0 \in \mathcal{C}_R$, the solution corresponding to $x_0$ and to the sampled-data feedback control law

$$u(t) = F(x_{t_k}), \quad t_k \leq t < t_{(k+1)}, \quad k = 0, 1, \ldots,$$

exists $\forall t \geq 0$ and, furthermore, satisfies:

$$x_t \in \mathcal{C}_E, \ \forall t \geq 0; \quad x_t \in \mathcal{C}_r, \ \forall t \geq T$$
Theorem 11. (*P., SICON 2014*) Let there exist a positive real $S$, a functional $V : \mathcal{C}_S \rightarrow R^+$, a map $k : \mathcal{C}_S \rightarrow R^m$ (continuous or not) such that:

i) $V$ is a CLKF in $\mathcal{C}_S$;

ii) $k$ is a steepest descent feedback induced by $V$, in $\mathcal{C}_S$.

Then, the steepest descent feedback $k$ stabilizes the system described by the RFDE in the sample-and-hold sense, in $\mathcal{C}_Q$, where $Q$ is a positive real satisfying the inequality $\alpha_1(S) > \alpha_2(Q)$, with

$$\alpha_1(s) = \eta e^{-\mu \Delta} p \circ \beta_1(s), \quad \alpha_2(s) = \gamma_2(s) + \eta p \circ \beta_2(s), \quad s \geq 0.$$
Corollary 12. Let there exist a diffeomorphism \( \Psi : \Omega_x \rightarrow \Omega_z \), with \( \Omega_x, \Omega_z \in \mathbb{R}^n \) open, bounded neighborhoods of the origin, functions \( \gamma_\psi, \bar{\gamma}_\psi \), of class \( K_\infty \), a Hurwitz matrix \( F \in \mathbb{R}^{n \times n} \), a positive real \( S \), a Lipschitz feedback \( k : \mathcal{C}_S \rightarrow \mathbb{R}^m \), zero at zero, such that: \( B_S \subset \Omega_x \);

\[
\gamma_\psi(|x|) \leq |\Psi(x)| \leq \bar{\gamma}_\psi(|x|), \quad \forall x \in \Omega_x;
\]

\[
\left. \frac{\partial \Psi(x)}{\partial x} \right|_{x=\phi(0)} f(\phi, k(\phi)) = F \Psi(\phi(0)), \quad \forall \phi \in \mathcal{C}_S
\]

Then, there exists a positive real \( Q \) such that the feedback \( k : \mathcal{C}_S \rightarrow U \) stabilizes in the sample-and-hold sense, in \( \mathcal{C}_Q \), the system described by the RFDE.
Human Glucose-Insulin System. Delays occur because of the reaction time of the pancreas to plasma-glucose variations.
\[
\frac{dG(t)}{dt} = -K_{xg} G(t) I(t) + \frac{T_{gh}}{V_G},
\]
\[
\frac{dI(t)}{dt} = -K_{xi} I(t) + \frac{T_{iG\text{max}}}{V_I} h(G(t - \tau_g)) + v(t),
\]
\[
G(\tau) = G_0, \quad I(\tau) = I_0, \quad \tau \in [-\tau_g, 0], \quad (2)
\]

- \( G(t) \) \([mM]\) plasma glucose concentration
- \( I(t) \) \([pM]\) plasma insulin concentration
The nonlinear map $h(\cdot)$ models the endogenous pancreatic insulin delivery rate as

$$h(G') = \frac{(\frac{G}{G^*})^\gamma}{1 + (\frac{G}{G^*})^\gamma},$$

where $\gamma$ is the progressivity with which the pancreas reacts to circulating glucose concentrations and $G^*$ is the glycemia at which the insulin release is half of its maximal rate. The control input, $v(t)$, is the exogenous intra-venous insulin delivery rate.
Sample-and-hold stabilizer

Let $G_{ref}$ be a positive constant, safe level of glycemia. Let $I_{ref}$ and $v_{ref}$ be the positive reals such that $(G_{ref}, I_{ref})$ is an equilibrium point for the glucose-insulin system described by the RFDE, forced by the constant input $v(t) = v_{ref}$. The RFDE can be rewritten with the new variables $x(t) = \begin{bmatrix} G(t) - G_{ref} \\ I(t) - I_{ref} \end{bmatrix}$ and with the new input $u(t) = v(t) - v_{ref}$. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, as

$$\Psi(x) = \begin{bmatrix} x_1 \\ -K_{xgi}(x_1 + G_{ref})(x_2 + I_{ref}) + K_{xgi}G_{ref}I_{ref} \end{bmatrix}. \quad (4)$$

Let $\Omega_x, \Omega_z$ be open neighborhoods of the origin such that the map $\Psi : \Omega_x \to \Omega_z$ defined, for $x \in \Omega_x$, as $\Psi(x) = \Psi(x)$, is a diffeomorphism.
Let us consider the state feedback $k : \mathcal{C} \rightarrow \mathcal{R}$ defined, for

$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in \mathcal{C}$, as (Palumbo, P., Panunzi, De Gaetano, DCD$\textit{S}$-B 2009)

$$k(\phi) = \begin{cases} 
-v_{\text{ref}} + \frac{\mathcal{P}(\phi_1(0)+G_{\text{ref}},\phi_2(0)+I_{\text{ref}},\phi_1(-\tau_g)+G_{\text{ref}}) - R\Psi(\phi(0))}{K_{xgi}(\phi_1(0)+G_{\text{ref}})} & , \\
\quad \phi_1(0) \neq -G_{\text{ref}}, \\
-v_{\text{ref}}, & \phi_1(0) = -G_{\text{ref}},
\end{cases}$$

(5)

where $\mathcal{P} : \mathcal{R}^3 \rightarrow \mathcal{R}$ is defined, for $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}^T \in \mathcal{R}^3$ as

$$\mathcal{P}(y_1, y_2, y_3) = -K_{xgi}y_2 \left( -K_{xgi}y_1y_2 + \frac{T_{gh}}{V_G} \right)$$

$$-K_{xgi}y_1 \left( -K_{xi}y_2 + \frac{T_{iG_{\text{max}}}}{V_I} h(y_3) \right)$$

(6)
Since the exogenous intra-venous insulin delivery rate cannot be negative, we have in this case that the input $v(t)$ must belong to the following set $\overline{V} = [0, v_{max}]$, where $v_{max}$ is a suitable positive real. It follows that $u(t)$ (and thus $k(\phi)$) must belong to the set $U = [−v_{ref}, v_{max} − v_{ref}]$. Since
\[
P(G_{ref},I_{ref},G_{ref}) \frac{k_{xgi}G_{ref}}{k_{xgi}G_{ref}} = v_{ref} > 0,
\]
taking into account that $\Psi(0) = 0$, it follows that there exists a positive real $S$ such that, for all $\phi \in C_S$, $k(\phi) \in U$. 
Digital Implementation of Glucose Controller


There exists a positive real $Q$ such that the state feedback $k$ stabilizes in the sample-and-hold sense, in $C_Q$, the glucose insulin-system.

The piece-wise constant control law $v(t)$ for the glucose-insulin system is defined as follows, for $t \geq 0$,

$$v(t) = \frac{\mathcal{P}(G(t_k), I(t_k), G(t_k - \tau_g)) - R\Psi \left( \begin{bmatrix} G(t_k) - G_{ref} \\ -K_{xgi}G(t_k)I(t_k) + \frac{Tgh}{V_G} \end{bmatrix} \right)}{K_{xgi}G(t_k)},$$

$$t_k \leq t \leq t_{(k+1)}, \ k = 0, 1, \ldots, \ t_0 = 0$$

(7)
A case of severe hyperglycemia (establishment of a state of frank Type 2 Diabetes Mellitus) is considered in Palumbo, P., Panunzi, De Gaetano, DCDS 2009. The delay $\tau_g$ is equal to 24 min.
Figure 3: Evolution of the plasma glycemia $G(t)$, with sampling period $\delta = 5 \text{ min}$
Figure 4: Evolution of the insulin $I(t)$, with sampling period $\delta = 5 \text{ min}$
Figure 5: Control Signal, with sampling period $\delta = 5 \text{ min}$
**Definition 14.** *(P., CDC 2015)* We say that a locally bounded state feedback $G : \mathcal{C} \to \mathbb{R}^m$ (continuous or not) is a linearizing virtual stabilizer for the nonlinear RFDE system, if there exist a non-negative integer $\omega$, non-negative reals $\Delta_j$, $j = 0, 1, \ldots, \omega$, with $0 = \Delta_0 < \Delta_1 < \cdots < \Delta_\omega = \Delta$, matrices $A_j \in \mathbb{R}^{n \times n}$, $j = 0, 1, \ldots, \omega$, such that, for any $\phi \in \mathcal{C}$, the equality holds, for the map $f$ describing the dynamics of the nonlinear RFDE system,

$$f(\phi, G(\phi)) = \sum_{j=0}^{\omega} A_j \phi(-\Delta_j),$$

and the linear time-delay system described by the equation

$$\dot{\xi}(t) = \sum_{j=0}^{\omega} A_j \xi(t - \Delta_j), \quad \xi_0 \in \mathcal{C},$$

is 0-GAS.
\[ \dot{x}(t) = -x(t) + 2x(t - \Delta) + |x(t)|u(t) \]

In this case, the map \( f \) is defined, for \( \phi \in C, u \in \mathbb{R} \), as
\[ f(\phi, u) = -\phi(0) + 2\phi(-\Delta) + |\phi(0)|u. \]

The discontinuous map \( G \) defined, for \( \phi \in C \), as \( G(\phi) = -2sgn(\phi(0)) \), is a linearizing virtual stabilizer. It seems hard (maybe impossible) to find out a continuous map such that the same goal of linearization and (virtual) stabilization is achieved.

Corresponding linear system:
\[ \dot{\xi}(t) = -3\xi(t) + 2\xi(t - \Delta) \quad (8) \]
Lemma 15. (Kharitonov, Zhabko, AUT 2003) If $k : C \to \mathbb{R}^m$ is a linearizing virtual stabilizer, then there exists a function $U : [-\Delta, \Delta] \to \mathbb{R}^{n \times n}$ with the following properties:

i) $U(0)$ is a symmetric positive definite matrix in $\mathbb{R}^{n \times n}$;

ii) $U$ is continuous in $[-\Delta, \Delta]$ and continuously differentiable in $[-\Delta, 0) \cup (0, \Delta]$, with

$$
\lim_{\tau \to 0^+} \frac{dU(\tau)}{d\tau} = \lim_{\tau \to 0^-} \frac{dU(\tau)}{d\tau} - I_n;
$$

(9)
iii) for the functional $W_U : C \rightarrow R^+$ defined, for $\phi \in C$, as

$$W_U(\phi) = \phi^T(0)U(0)\phi(0) + 2\phi^T(0) \sum_{j=1}^{\bar{\omega}} \int_{-\Delta_j}^{0} U(-\theta - \Delta_j)A_j\phi(\theta)d\theta$$

$$+ \sum_{i=1}^{\bar{\omega}} \sum_{j=1}^{\bar{\omega}} \int_{-\Delta_i}^{0} \phi^T(\theta_1)A_i^T \left( \int_{-\Delta_j}^{0} U(\theta_1 + \Delta_i - \theta_2 - \Delta_j)A_j\phi(\theta_2)d\theta_2 \right) d\theta_1$$

$$+ \sum_{j=1}^{\bar{\omega}} \int_{-\Delta_j}^{0} (1 + \Delta_j + \theta)\phi^T(\theta)\phi(\theta)d\theta$$

the following inequalities hold, for suitable positive reals $a_i$, $i = 1, 2,$

$$a_1|\phi(0)|^2 \leq W_U(\phi) \leq a_2\|\phi\|_\infty^2,$$

$$D^+W_U(\phi, k(\phi)) \leq -\left| \begin{bmatrix} \phi^T(0) & \phi^T(-\Delta_1) & \cdots & \phi^T(-\Delta_\omega) \end{bmatrix}^T \right|^2$$
Theorem 16. (P., CDC 2015) Any linearizing virtual stabilizer $k : C \rightarrow R^m$ is a stabilizer in the sample-and-hold sense.
The linear case

\[ \dot{x}(t) = \sum_{j=1}^{p} A_j x(t - \Delta_j) + B u(t), \quad x_0 \in \mathcal{C}, \quad (10) \]
Corollary 17. (P., CDC 2015) Let there exist \((p + 1)\) matrices \(K_j \in \mathbb{R}^{m \times n}, j = 0, 1, \ldots, p\), such that the closed-loop system with

\[
u(t) = K \left[ x^T(t) \ x^T(t - \Delta_1) \ \cdots \ x^T(t - \Delta_p) \right]^T,
\]

\[
K = \left[ \begin{array}{cccc}
K_0 & K_1 & \cdots & K_p \\
\end{array} \right],
\]

is 0-GAS. Then, the feedback \(k : \mathcal{C} \to \mathbb{R}^m\), defined, for \(\phi \in \mathcal{C}\), as

\[
k(\phi) = K \left[ \phi^T(0) \ \phi^T(-\Delta_1) \ \cdots \ \phi^T(-\Delta_p) \right]^T,
\]

is a stabilizer in the sample-and-hold sense for the linear system.
Work in Progress and Future Developments

- Sampled-data observer-based (continuous time) controllers for systems described by RFDEs.

- Sample-and-hold stabilizers for nonlinear systems with time-varying time-delays.

- Stabilization in the sample-and-hold sense of systems described by RFDEs with discontinuous right-hand side.

- Robustness with respect to actuation disturbances and observation errors.
I wish to express my gratitude to Hiroshi Ito for kindly inviting me to deliver this talk.

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