Fractional Order Systems

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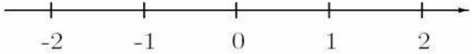
Generalization of Number System

. . . from integer to non-integer . . .



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

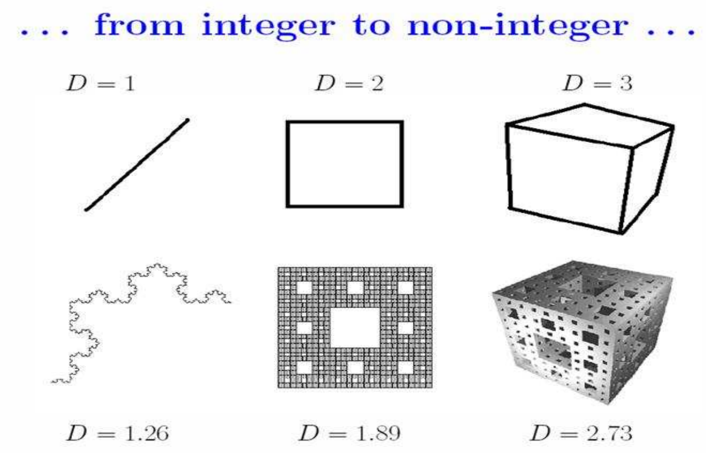
$$x^n = e^{n \ln x}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot n = n!$$

Generalization of Dimension



Generalization of Calculus

Interpolation of operations

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

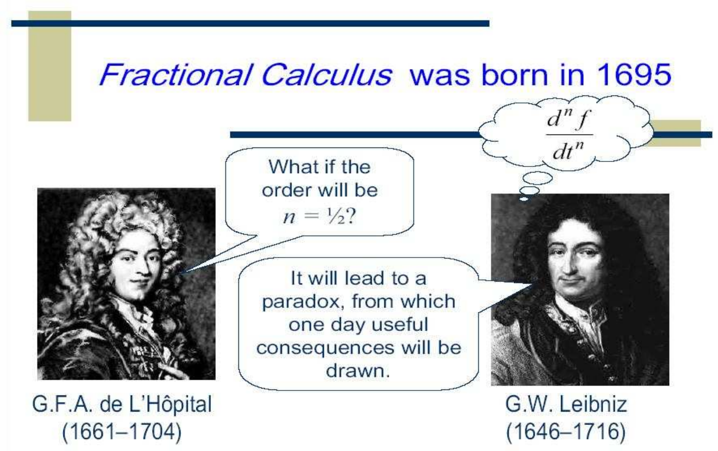
$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$

Fractional Order Thinking or In Between Thinking

Examples:

- **Between integers there are non-integers**
- **Between integer dimensions, there are fractal dimensions**
- **Between logic 0 and logic 1, there is the fuzzy logic**
- **Non-Integer order calculus or fractional order calculus (abuse of terminology)**

Origin of Fractional Calculus



Origin of Fractional Calculus

Riemann–Liouville definition

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ (n-1 \le \alpha < n) $$

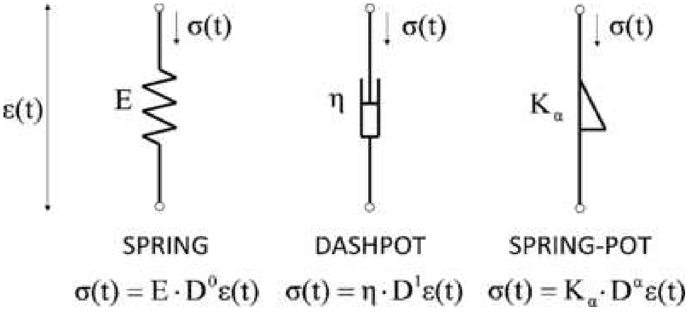
G.F.B. Riemann (1826–1866)

J. Liouville (1809–1882)

Biorheology and Fractional Calculus

Biorheology

study of flow properties (rheology) of biological fluids



Tissue-like materials (polymers, gels, emulsions, composites and suspensions) exhibit power-law responses to an applied stress or strain.

Where to Begin?

Cauchy's integral and power-law function

Consider an antiderivative of the function $f(t)$, $D^{-1}f(t)$, then

$$D^{-1}f(t) = \int_0^t f(x)dx$$

Now perform the repeated application of the antiderivative operator

$$D^{-2}f(t) = \int_0^t \int_0^\tau f(\tau)d\tau dx$$

Cauchy's Integral and Power-law Function

taking into account the $x - \tau$ plane, one can reverse the sequence of integrations

$$D^{-2}f(t) = \int_0^t \int_\tau^t f(\tau)dxd\tau$$

As $f(\tau)$ is a constant with respect to x, one can write

$$D^{-2}f(t) = \int_0^t (t-\tau)f(\tau)d\tau$$

Similarly

$$D^{-n}f(t) = \underbrace{\int \ldots \int_0^t}_{n} f(\tau)\underbrace{d\tau \ldots d\tau}_{n} = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau$$

Factorial & Gamma functions

Can we "find a smooth curve that connects the points (n,n!)"?

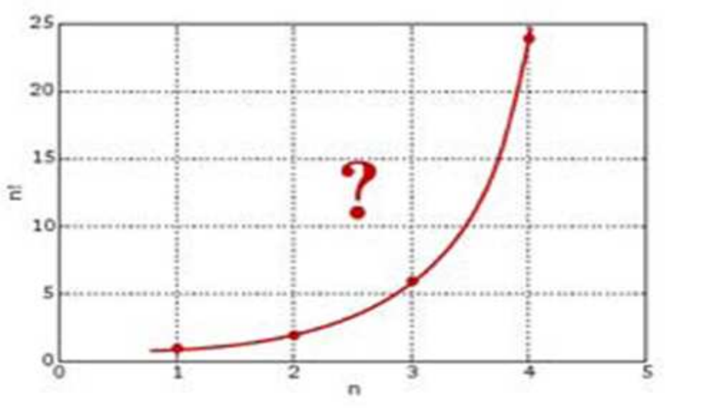


Figure: Evolution of (n,n!)

The Gamma function

Definition:

$$\Gamma(z) =: \int_0^{\infty} e^{-t} t^{z-1} dt$$

Properties

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{z+1-1} dt = \int_0^{\infty} e^{-t} t^{z} dt$$

$$= [-e^{-t} t^{z}]_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z)$$

Origin of Riemann-Liouville's definition

Riemann Liouville fractional order integral (generalization for $n \in \mathbb{R}^+$

$$D^{-n}f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau$$

Riemann Liouville fractional-order derivative of order $\alpha \in R^+$ and $m-1 < \alpha < m$, $m \in \mathbb{N}$ has the following form

$$_R D^{\alpha} f(t) = D^m D^{-(m-\alpha)} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau\right]$$

Fractional-order Derivatives continued...

An alternative definition for the fractional-order derivative was introduced by Caputo as

$$ {}_{C}D^{\alpha} f(t) = D^{-(m-\alpha)} D^{m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{m}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau $$

Derivative of constant $f(t) = 1$ using Riemann-Liouville definition

$$ {}^{RL}_{t_0}D_t^{\frac{1}{2}} 1 = \frac{d}{dt}\left(\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{1/2}.1dt\right) $$
$$ = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\left(\left[-2(t-\tau)^{1/2}\right]_0^t\right) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\left(2\sqrt{t}\right) = \frac{1}{\sqrt{\pi}\sqrt{t}} $$

Fractional-order Derivatives continued...

Compare the following two systems for all $0 < \lambda < 1$ and initial condition $x(0) = x_0$

$$\frac{d}{dt}x(t) = \lambda t^{\lambda-1} \Rightarrow x(t) = t^{\lambda} + x(0)$$

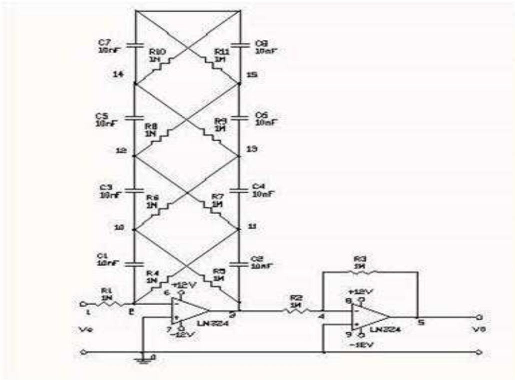
$$D^{\alpha}x(t) = \lambda t^{\lambda-1}; 0 < \alpha < 1 \Rightarrow x(t) = \frac{\lambda\Gamma(\alpha)t^{\lambda+\alpha-1}}{\Gamma(\lambda+\alpha)} + x(0)$$

Observations from above example:

- The integer order system is unstable for all $\lambda \in]0, 1[$
- The fractional order system is stable for all $\lambda \in]0, 1-\alpha[$

Conclusion: different characteristics (not local due to history of the integral)

Realizations of Fractional Operators



Analog $1/\sqrt{s}$ using op-amps.

Conclusion

Fractional calculus is the calculus of twenty-first century for physical system description and controls.

Thank you for your attention