CERTAIN CONCEPTS ON FIRST ORDER SLIDING MODE

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Consider autonomous systems of the form

$$\dot{x}(t) = Ax(t) \tag{1}$$

where $x(t) \in \mathbb{R}$ and $A \in \mathbb{R}$ to formulate the **concepts of algorithm.**

Traditionally, the system matrix A is considered as a matrix, i.e $A \in \mathbb{R}^{n \times n}$ and $x(t) \in \mathbb{R}^n$ and if the Eigen values of matrix A are in the left half s-plane then one can easily prove that the system is asymptotically stable.

Now, let us consider a forced system (or system with control) of the following form

$$\dot{x}(t) = Ax(t) + Bu, \quad x(t) \in \mathbb{R}, \quad A \in \mathbb{R}, \quad B \in \mathbb{R}, \quad u \in \mathbb{R} \tag{2}$$

- Our main objective is to design a control u, so that (2) is converted to an autonomous system like (1).
- If we are able to do this, then one can claim that the system (2) has similar properties as (1).
- In other words, one can say that (1) is an **algorithm**.
- **Therefore, here the meaning of algorithm is a set of autonomous differential equations/equation which have the desired properties.**

Algorithm Vs Control

- The simplest form of state dependent controller for the system (2) is $u = kx$.
- With this control the closed loop system (2) is $\dot{x}(t) = (A + Bk)x(t)$.
- Therefore, now we have to select k such that the new system matrix $(A + Bk)$ is Hurwitz.

Remark:

- Any non autonomous system is converted to a specific algorithm, which have the desired properties, by substituting an appropriate form of a control.
- Therefore, the controller design is always dependent on the algorithm.
- **Hence, the algorithm is the most basic element of any control system design.**
- Therefore the motivation behind this talk is to give more emphasis to the algorithm than the control.

- In the above analysis, we are considering only nominal systems without any external disturbances/perturbations.
- But almost all practical systems are influenced by external disturbances/perturbations.

Let a perturbed autonomous system be represented as

$$\dot{x}(t) = Ax(t) + d \tag{3}$$

where states $x(t) \in \mathbb{R}^n$, system Hurwitz matrix $A \in \mathbb{R}^{n \times n}$.

Assumptions:-

- The disturbance $d = d_1 + d_2 x(t) \in \mathbb{R}^{n \times 1}$.
- d_1 is the non vanishing disturbance which is bounded as $d_{1L} \leq \|d_1\| \leq d_{1U}$.
- d_2 is the vanishing disturbance which is bounded as $d_{2L} \leq \|d_2\| \leq d_{2U}$.

- Before going to further analysis, let us define the types of disturbances that frequently affects any deterministic systems.
- Deterministic systems represent that class of system where the disturbance affecting is bounded, but the actual value of disturbance is unknown.

Vanishing Disturbance:

- The disturbance which vanishes at the equilibrium point(origin) i.e, if $d = 0$, at equilibrium point, then the disturbance is called vanishing disturbance.
- In such a case the equilibrium point of perturbed system (3) is same as that of the nominal system (1).
- This means equilibrium point is preserved and one can analyze the stability with respect to the origin.

Non-vanishing Disturbance:

- If $d \neq 0$, at equilibrium point(origin), then the disturbance is called non-vanishing disturbance.
- In this case equilibrium point of perturbed system (3) is not same as the nominal system (1).
- Therefore, one can no longer study the problem as a question of stability of equilibria.
- Some new formulation is required to study these classes of systems.

Lyapunov

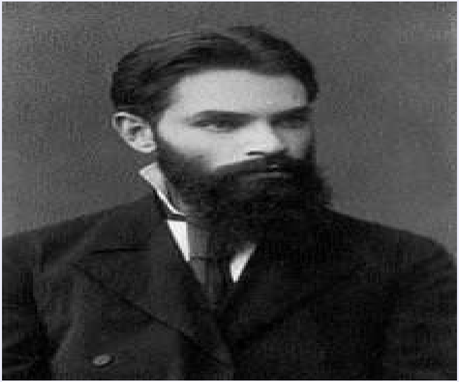

Figure : Father of Stability Theory:-Lyapunov

- There are several ways of analyzing the stability of the dynamical systems, Lyapunov stability is one among them.
- In Lyapunov stability an energy like function is defined,
- which is always positive and whose value is zero at the equilibrium point and also, derivative of this energy like function is monotonically or at least nonincreasing along the system trajectory.

Mathematically, one can write

$$\begin{aligned} V(x) &> 0 \\ V(x = 0) &= 0 \\ \dot{V}(x) &< 0 \end{aligned} \tag{4}$$

where, $V(x)$ is the positive definite energy like function chosen.

Remark:

Lyapunov function might also be explicitly time dependent. But in sliding mode we are considering only state dependent Lyapunov functions.

The most frequently used Lyapunov candidate function is a quadratic Lyapunov function. The quadratic Lyapunov function for the system (3) is given by

$$V(x) = x^T P x \tag{5}$$

where $P = P^T$ is a positive definite symmetric matrix.

Taking the time derivative of the Lyapunov function along the system trajectories(3), we obtain,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x + d^T P x + x^T P d \\ &= x^T (A^T P + PA) x + d^T P x + x^T P d \end{aligned} \tag{6}$$

Let $Q = Q^T > 0$ be another positive definite symmetric matrix such that,

$$A^T P + PA = -Q \tag{7}$$

Using the Cauchy-Schwartz inequality

$$
\begin{aligned}
|x^T P d| &\leq \lambda_{max}(P)\|x\|\|d\| \\
|d^T P x| &\leq \lambda_{max}\|x\|\|d\|
\end{aligned} \tag{8}
$$

where $\|.\|$ represents the norm

and Rayleigh inequality

$$
\begin{aligned}
\lambda_{min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{max}(P)\|x\|^2 \\
\lambda_{min}(Q)\|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|^2
\end{aligned} \tag{9}
$$

where $\lambda_{min}(.)$ and $\lambda_{max}(.)$ represent the minimum and maximum eigenvalues of the matrices respectively. Now,

$$
\begin{aligned}
\dot{V}(x) &= x^T(-Q)x + d^T P x + x^T P d \\
&\leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)\|x\|\|d\|
\end{aligned} \tag{10}
$$

Let us consider the case of vanishing disturbance, $d = d_2 x$ or $\|d\| \leq d_{2U}\|x\|$. Substituting the value of $\|d_2\|$ in Eqn.(10)

$$\dot{V}(x) \leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)d_{2U}\|x\|^2$$
$$\leq -(\lambda_{min}(Q) - 2\lambda_{max}(P)d_{2U})\|x\|^2 \quad (11)$$

Hence origin is globally asymptotically stable if $d_{2U} < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$ which satisfies the condition $\dot{V}(x) < 0$.

Now consider the case of non vanishing disturbance $d = d_1$ or $\|d\| \leq d_{1U}$. Substituting the value of $\|d_1\|$ in Eqn.(10)

$$\dot{V}(x) \leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)d_{1U}\|x\|$$
$$\leq -(\lambda_{min}(Q)\|x\| - 2\lambda_{max}(P)d_{1U})\|x\| \quad (12)$$

Hence system trajectories are **ultimately bounded with respect to the set** $\|x(t)\| < \frac{2\lambda_{max}(P)d_{1U}}{\lambda_{min}(Q)}$ for satisfying the condition $\dot{V}(x) < 0$.

From the above analysis it is clear that non vanishing disturbances has to be deal in different way.

But one can argue that, the system can be stabilized easily with the help of a control variable.

- Now, let us suppose that we have an control input also in the system $\dot{x}(t) = Ax(t) + d$.
- For simplicity, we consider a system with the number of states variables equal to the number of control inputs or mathematically $B \in \mathbb{R}^{n \times n}$ and $u \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t) + Bu + d \tag{13}$$

It is shown in literature that even with the stated condition, stabilization of the system (13) is not easy, with a simple continuous controller.

Some More Analysis of Systems with a Non-vanishing Disturbance

Suppose, we are taking a state feedback based control assuming that all the state variables are available and the system is completely controllable.

After applying the state feedback control $u = Kx(t)$, the closed loop system (13) becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t) + d \\ &= (A + BK)x(t) + d \end{aligned} \tag{14}$$

Eqn.(14) is again same as (3) with a non vanishing disturbance and again similar kind of situation arises and it is difficult to stabilize the system at the origin.

This problem remained unsolved until the **high gain feedback** theory was published in the literature.

In high gain feedback, the control is designed as $u = \lim_{\tau \to 0} \frac{1}{\tau} Kx(t)$, where $K < 0$ and $\tau \to 0^+$.

After applying high gain feedback the closed loop system (13) can be represented as

$$\dot{x}(t) = Ax(t) + \lim_{\tau \to 0} \frac{BK}{\tau} x(t) + d \tag{15}$$

On simplification,

$$\tau \dot{x}(t) = \tau A x(t) + BKx(t) + \tau d \Rightarrow BKx(t) \approx 0 \tag{16}$$

because $\tau \dot{x}(t) = \tau A x(t) = \tau d \approx 0 \Rightarrow x \to 0$ if BK is invertible.

Observations:

- Hence, mathematically one can see that inspite of non vanishing disturbances, the system can be stabilized at the origin.
- But this control has infinite magnitude, therefore practically it is not feasible to apply this type of control because of the finite bandwidth of all practical actuators.
- Again, problem of non-vanishing disturbance remains open in the practical point of view, but at least a good mathematical insight is developed.

High Gain Feedback for More Generalized System

Consider the following system

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t)$$
$$\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + u + \Psi_0 \quad (17)$$

High gain feedback control is designed as

$u = -A_{21}z_1(t) - A_{22}z_2(t) + \lim_{\tau\to 0}\frac{1}{\tau}(K_1z_1(t) + K_2z_2(t))$ for the system (17)

After applying high gain feedback the closed loop system (17) can be represented as

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t)$$
$$\dot{z}_2(t) = \lim_{\tau\to 0}\frac{1}{\tau}(K_1z_1(t) + K_2z_2(t)) + \Psi_0 \quad (18)$$

One can further write

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t)$$
$$\lim_{\tau\to 0}\tau\dot{z}_2(t) = (K_1z_1(t) + K_2z_2(t)) + \lim_{\tau\to 0}\tau\Psi_0 \quad (19)$$

$$\Rightarrow \dot{z}_1(t) = A_{11} z_1(t) + A_{12} z_2(t)$$
$$K_1 z_1(t) + K_2 z_2(t) \approx 0 \tag{20}$$

$$\Rightarrow \dot{z}_1(t) \approx A_{11} z_1(t) - \frac{A_{12}K_1}{k_2} z_1(t)$$
$$\approx \left(A_{11} - \frac{A_{12}K_1}{k_2} \right) z_1(t) \tag{21}$$

which implies $z_1 \approx 0$ if poles of $A_{11} - \frac{A_{12}K_1}{k_2}$ lies in left half plane.

Since $K_1 z_1(t) + K_2 z_2(t) \approx 0$ and $z_1(t) \approx 0$, which further implies $z_2(t) \approx 0$.

- It is also observed that after applying such a high gain feedback control the system order is reduced, which is clearly seen from Eqn.(16).
- Vast amount of research was carried out by many, to explore the possibility of having same phenomenon like the above with a finite magnitude of control.
- The only unexplored area was controlling systems using discontinuous feedback controllers which will expand the scope of selecting the controllers.
- Enormous research in this direction led to the development of a discontinuous algorithm for the discontinuous autonomous system which can be stabilized at the origin inspite of non vanishing disturbances.

Discontinuous autonomous systems without disturbance

Let us take the case of the following simplest disturbance free one dimensional discontinuous autonomous system

$$\dot{x}(t) = -k\text{sign}(x(t)) \tag{22}$$

where $x(t) \in \mathbb{R}$, $k > 0$ and

$$\text{sign}(x(t)) = \begin{cases} +1 & \text{if } x(t) > 0 \\ \{-1 \quad 1\} & \text{if } x(t) = 0 \\ -1 & \text{if } x(t) < 0 \end{cases} \tag{23}$$

Remark:

- One can observe that Eqn.(22) is not a simple differential equation,
- but it contains infinite number of differential equations because of the infinite number of possibility at $x(t) = 0$ (differential equation of right hand side can take any value between {-1 1}).
- This type of equations are named as differential equations with discontinuous right hand side or more soundly **differential inclusion**.
- The existence of solution of this type of differential equation with discontinuous right hand side is not explored in the classical sense.

Discontinuous autonomous systems without disturbance

Taking the simplest candidate Lyapunov $V = \frac{1}{2}x^T(t)x(t)$ for the system (22) and taking time derivative of Lyapunov function along the system trajectory, we obtain

$$\begin{aligned} \dot{V} &= x(t)\dot{x}(t) \\ &= x(t)(-k\text{sign}(x(t))) \\ &= -k|x(t)| < 0 \end{aligned} \tag{24}$$

because $x(t)\text{sign}(x(t)) = |x(t)|$.

- Hence, one can conclude from the above analysis that system (22) is asymptotically stable at the origin.
- But, we can also see that the above analysis can be extended like

$$\dot{V} = -k(2V)^{\frac{1}{2}} \tag{25}$$

because $V = \frac{1}{2}(x(t))^2 = \frac{1}{2}(|x(t)|)^2$.

One can solve Eqn.(25) as

$$\frac{dV}{dt} = -k(2V)^{\frac{1}{2}} \Rightarrow \int_{V(0)}^{0} \frac{dV}{(V)^{\frac{1}{2}}} = -k2^{\frac{1}{2}} \int_{0}^{T} dt$$

$$\frac{(V(0))^{\frac{1}{2}}}{\frac{1}{2}} = k2^{\frac{1}{2}} T \Rightarrow T = \frac{(2V(0))^{\frac{1}{2}}}{k} \tag{26}$$

where $V(0)$ is initial value of Lyapunov function and T is final convergence time to the origin.

Note:

- We have already proved that, the system is asymptotically stable, means that the system states are attracted towards the origin,
- therefore the value of Lyapunov function also decreases along the trajectory,
- hence we are taking the lower limit of integration $V = V(0)$ at $t = 0$ and finally system converges towards the origin so the upper limit of integration is $V = 0$ at $t = T$.

Remark:

- From the above analysis it is clear that the system states converge to the origin in a finite time $T = \frac{(2V(0))^{\frac{1}{2}}}{k}$.
- Therefore, the system (22) is finite time stable which has a quick convergence than an asymptotic stable system like (1).

- Now we come to our main problem of interest, system (22) with a non-vanishing bounded disturbance $|d| \leq d_{max} \in \mathbb{R}$ where d_{max} is the maximum value of the disturbance magnitude.
- Our task is to analyze the stability of the perturbed system, which is expressed as

$$\dot{x}(t) = -k\text{sign}(x(t)) + d \tag{27}$$

Again choosing the same candidate Lyapunov $V = \frac{1}{2}x^T(t)x(t)$ for the system (27) and taking its time derivative along the system trajectory, we can write

$$\dot{V} = x(t)\dot{x}(t) = x(t)(-k\text{sign}(x(t)) + d) = -k|x(t)| + dx(t) \tag{28}$$

Now using norm inequality,

$$dx(t) \leq |d||x(t)| \leq d_{max}|x(t)| \tag{29}$$

Using Eqn.(28) and (29),

$$\begin{aligned} \dot{V} &\leq -k|x(t)| + d_{\max}|x(t)| \\ &\leq -(k - d_{\max})|x(t)| \\ &\leq -\eta(2V)^{\frac{1}{2}} \end{aligned} \tag{30}$$

where $\eta \geq (k - d_{\max}) > 0$. Therefore, if $k \geq d_{max}$ then the system (27) is finite time stable. By comparison Lemma again one can calculate the time $T \leq \frac{(2V(0))^{\frac{1}{2}}}{\eta}$.

Consider a generic mechanical system given by following sets of differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f(t,x) \\ \sigma &= x_2 \end{aligned} \tag{31}$$

where $x_1, x_2, u, f(t,x)$ and σ are the position, velocity, control input, all other nonlinear dynamics with perturbation/unmodelled dynamics and the measured output respectively.

Now, if we select control as

$$u = -k\text{sign}(\sigma) \tag{32}$$

Above control is able to stabilized $x_2 = 0$ in finite time same as (27), mathematically when $x_2 = 0 \Rightarrow x_1 = c$, where c is an arbitrary constant.

The Open Problems are

- The system is stopped, but where?
- No control over x_1 (position).
- Can we manipulate both x_1 and x_2 at the same time?
- Definition of solution at the discontinuity surface is needed.
- High frequency discontinuous (switching) control
- Chattering.

Problem Formulation

Design $u(t)$ such that

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0 \tag{33}$$

inspite of bounded uncertainty, i.e, $|f(t,x)| < L$, where $f(t,x)$ represents the modeling imperfections and external perturbations and L represents the maximum bound of uncertainty.

Remarks:-

- As we already discussed that system of the form $\dot{x}(t) = u + d$, where $u = -k\text{sign}(x(t)) \in \mathbb{R}$ is able to stabilize x only when $x \in \mathbb{R}$.
- But in practical scenarios almost all systems have more number of states than number of input.
- If number of free variables to be control is equal to number of controlled inputs then the above proposed controller will work well.
- But, when one can try to apply the same algorithm for the vector systems with a scalar control, then further concepts of manifold is needed. The required manifold in this case is known as **sliding surface**.
- Therefore, sliding manifold is one of the ways by which we can control any vector system using scalar control.

Founder of Sliding Mode

Prof. Utkin and Prof. Emel'yanov, IFAC Sensitivity Conference, Dubronovik 1964

Figure : Father of Sliding Mode Control Theory

- Although, a k-dimensional manifold in $\mathbb{R}^n (1 \leq k < n)$ has a rigorous mathematical definition.
- But for our purpose, it is sufficient to think of a k-dimensional manifold as the solution of the equation

$$\sigma(x) = 0$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is sufficiently smooth (that is, sufficiently many times continuously differentiable).

Example:

The unit circle

$$\left\{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \right\}$$

is a 1-dimensional manifold in $\mathbb{R}^2$.
Example: The unit sphere

$$\left\{ x \in \mathbb{R}^n \mid \sum_1^n x_i^2 = 1 \right\}$$

is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$.

Example:-Consider the following uncertain chain of integrators

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(x) + g(x)u \end{aligned} \tag{34}$$

- Here, $\sigma(x) = \sum_{i=1}^{n} c_i x_i = 0$, (where $c_i > 0, i = 1, 2 \cdots n$ are the coefficients of the Hurwitz matrix) is $(n-1)$ dimensional manifold in $\mathbb{R}^n$.
- In the first order sliding mode $\sigma(x) = 0$ is commonly taken as a manifold.
- But manifold can be nonlinear also.

Let us now define a sliding manifold.

Definition of Sliding Manifold

Chosen line/plane/surface of the state space along which the motion of the system trajectory occur after finite time is known as sliding manifold.

Consider the nonlinear uncertain system of the following form

$$\dot{x} = f(x) + g(x)u \tag{35}$$

where $x \in X \in \mathbb{R}^{n\times 1}$ the state vector and $u \in \mathbb{R}$ the control input.

Assumptions:

- Functions $f(x)$ and $g(x)$ are smooth uncertain functions and are bounded for X
- $f(x)$ contains unmeasured perturbation terms and $g(x) \neq 0$ for $x \in X$
- System (35) is controllable for all $x \in X$.
- Continuous function σ (named as sliding variable) admits a relative degree equal to 1 with respect to u.
- $|\Psi| \leq |\Psi_M|$ and $0 < \Gamma_m \leq \Gamma \leq \Gamma_M$ for $x \in X$. It is assumed that Ψ_M, Γ_m and Γ_M exist and known.

Aim -**The control objective consists in forcing the continuous function $\sigma(x, t)$, to 0 in finite time.**

Taking time derivative of sliding variable $\dot{\sigma}$ along the system (35), one can write

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x}\dot{x} + \frac{\partial \sigma}{\partial t} = \underbrace{\frac{\partial \sigma}{\partial x} f(x) + \frac{\partial \sigma}{\partial t}} + \frac{\partial \sigma}{\partial x} g(x)u$$

$$= \Psi(x,t) + \Gamma(x,t)u \tag{36}$$

where $\Psi(x,t) = \frac{\partial \sigma}{\partial x} f(x) + \frac{\partial \sigma}{\partial t}$ and $\Gamma(x,t) = \frac{\partial \sigma}{\partial x} g(x)$.

Definition:

- Consider the nonlinear system (35), and let the system be closed by some possibly dynamical discontinuous feedback.
- Variable σ is a continuous function, and the set

$$S = \{x \in X | \sigma(x,t) = 0\} \tag{37}$$

  called sliding surface, is non-empty and is locally an integral set in the Filippov sense,
- i.e. it consists of Filippov's trajectories of the discontinuous dynamical system.
- The motion on S is called sliding mode with respect to the sliding variable σ.

Remark :

In simple word one can define first order ideal sliding mode algorithm as finite time stabilization of uncertain bounded integrator (36) using discontinuous feedback (with infinite switching frequency) in the sense of Filippov.

Real Sliding Mode

Given the sliding variable $\sigma(x, t)$, the real sliding surface associated with (35) is defined as (with $\delta > 0$)

$$S^* = \{x \in X| \quad |\sigma| < \delta\}. \tag{38}$$

Definition :

- Consider the non-empty real sliding surface S^* given by (38),
- and assume that it is locally an integral set in the Filippov sense.
- The corresponding behavior of system (35) on (38) is called real sliding mode with respect to the sliding variable $\sigma(x, t)$.

Phases of Sliding Mode Control

- **Reaching phase**: Where the system state is driven from any initial state to reach the switching manifolds (the anticipated sliding modes) in finite time.
- **Sliding-mode phase**: Where the system is induced into the sliding motion on the switching manifolds, i.e., the switching manifolds become an attractor.

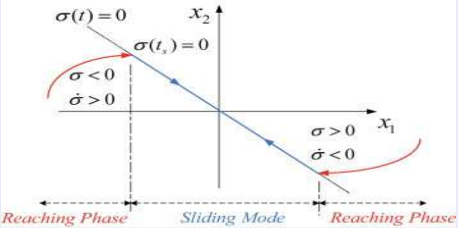

Figure : Phases of Sliding Mode Control

Design Steps of Sliding Mode Control

- **Switching manifold selection**: A set of switching manifolds are selected with prescribed desirable dynamical characteristics.
- **Discontinuous control design**: A discontinuous control strategy is formed to ensure the finite time reachability of the switching manifolds. The controller may be either local or global, depending upon specific control requirements.

Finite Time Reachability to the Sliding Manifold

- Consider the autonomous differential equation described by

$$\dot{x} = f(x) \tag{39}$$

  where, $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous on an open neighbourhood $\mathcal{D}$ of the origin and locally lipschitz on $\mathcal{D} \setminus 0$.
- Assuming the origin is the only equilibrium point of (39) following are some results for finite time stability of the origin given in next slide.

Finite Time Stability

Definition:

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ such that the following conditions hold.

1. V is positive definite.
2. $\dot{V}$ is continuous and negative on $\mathcal{D} \setminus 0$
3. There exists real positive numbers c and α such that $\dot{V} + cV^{\alpha} \leq 0$ on $\mathcal{D} \setminus 0$.

then the origin of (39) is finite time stable.

Above result is a special case of more general result available in literature as follows.

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ such that following conditions hold.

1. V is positive definite.
2. $\dot{V} \leq \phi(V)$, and using comparison principle $\dot{v} = \phi(v)$, with $v \in \mathbb{R}$ and $V(x_0) \leq v(0)$ is a finite time stable differential equation.

then the origin of (39) is finite time stable.

Synthesizing the reaching law:

- Let $V = \frac{1}{2}\sigma^2$ be the Lyapunov function for synthesizing the reaching law, where $\sigma \in \mathbb{R}$ is the sliding surface.
- The time derivative of the Lyapunov function is given as

$$\dot{V} = \sigma\dot{\sigma} \tag{40}$$

- For the asymptotic stability, $\dot{V} < 0 \Rightarrow \sigma\dot{\sigma} < 0$.
- But the objective here is to reach the sliding surface in finite time, for this purpose if $\dot{V}(x) + cV^{\alpha}(x)$ is negative semidefinite,
- then one can ensure the finite time reachability to the surface.
- There are infinite number of solutions satisfying the inequality $\dot{V}(x) + cV^{\alpha}(x) \leq 0$.
- One of the possibility is

$$\sigma\dot{\sigma} \leq -\eta|\sigma| \tag{41}$$

where $\eta > 0$ which is known as $\eta-$**reachability condition**, which ensures finite time convergence to $\sigma = 0$,

Condition for Convergence

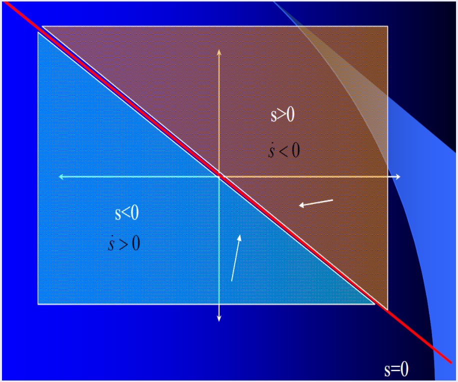

Figure : Condition for Convergence

- By integration of (41)

$$|\sigma(t)| - |\sigma(0)| \leq -\eta t \quad (42)$$

showing that the time required to reach the surface, starting from initial condition $\sigma(0)$, is bounded by

$$T = \frac{\sigma(0)}{\eta} \quad (43)$$

- This possibility comes into picture by putting $\alpha = \frac{1}{2}$ into the condition $\dot{V}(x) + cV^{\alpha}(x) \leq 0$.
- One of the solution (41) is $\dot{\sigma} = -k\text{sign}(\sigma)$, $k > 0$, which is known as **constant reaching law**.

Problem with constant reaching law

- The reaching rate is directly dependent on the magnitude of the gain.
- But, the main problem with a large gain is that, the chattering magnitude is directly proportional to the gain of the discontinuous term.

Problem with constant reaching law and simple modification

- If $k = k_1 + k_2|\sigma|$, $k_1 > 0, k_2 > 0$ the convergence speed of the trajectory towards the sliding surface,
- when the initial state is far away from the sliding surface is large due to the term $k_2|\sigma|$
- and also chattering magnitude is reduced because $k_2|\sigma| = 0$, at $\sigma = 0$.

The above modification in constant rate reaching law is mathematically written as

$$\begin{aligned}\dot{\sigma} &= -k\text{sign}(\sigma) \\ &= -(k_1 + k_2|\sigma|)\text{sign}(\sigma) = -k_2\sigma - k_1\text{sign}(\sigma)\end{aligned} \tag{44}$$

Eqn. (44) is known as **proportional rate reaching law**.
This satisfies the $\eta-$ reachability condition

$$\begin{aligned}\sigma\dot{\sigma} &= \sigma(-k_2\sigma - k_1\text{sign}(\sigma)) \\ &= -k_2\sigma^2 - k_1|\sigma| \leq -k_1|\sigma|\end{aligned} \tag{45}$$

- $\eta-$reachability is only a sufficient but not a necessary condition for the finite time reachability to the sliding surface.
- One can see the following power rate reaching law (46)which has the property of finite time reachability to the sliding surface but not satisfying the $\eta-$reachability condition.

$$\dot{\sigma} = -k|\sigma|^{\alpha}\text{sign}(\sigma), \quad 0 < \alpha < 1 \tag{46}$$

- Checking the $\eta-$reachability condition

$$\sigma\dot{\sigma} = -k|\sigma|^{1+\alpha} \nleq -k|\sigma| \tag{47}$$

when $0 < \sigma < 1$ then, $-k|\sigma|^{1+\alpha} \nleq -k|\sigma|$. But (46) is finite time stable.

For justifying this statement mathematically take the Lyapunov function as $V = |\sigma|$. Taking the time derivative of this Lyapunov function along the system (46), one can write

$$\dot{V} = \dot{\sigma}\text{sign}(\sigma) = -k|\sigma|^{\alpha} = -kV^{\alpha} \tag{48}$$

Hence the above claim is justified.

Advantages of Sliding Mode Control

Advantages:

- Exact compensation (insensitivity) with respect to bounded matched uncertainties.
- Reduced order dynamics during sliding.
- Finite-time convergence to the sliding surface.

Above advantages can be illustrated using the following example of single-input-single-output systems with motion equations in canonical space

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -\sum_{i=1}^{n} a_i(t)x_i + b(t)u + f(t)
\end{aligned}
\tag{49}
$$

where $a_i(t)$ and $b_i(t)$ are unknown parameters and $f(t)$ is an unknown disturbance. Here $x = x_1$ is a controlled variable and its time derivatives $x^{i-1} = x_i, i = 1, 2, \cdots, n$ are components of a state vector in the canonical space.).

For the system (49) switching manifold is designed as $\sigma(x) = \sum_{i=1}^{n} c_i x_i = 0$, (where $c_i > 0, i = 1, 2 \cdots n$ are the coefficients of the Hurwitz matrixNow taking the time derivative of sliding manifold one can write

$$\begin{aligned}\dot{\sigma} &= c_1\dot{x}_1 + c_2\dot{x}_2 + \cdots + c_n\dot{x}_n \\ &= c_1x_2 + c_2x_3 + \cdots + c_n\left(-\sum_{i=1}^{n} a_i(t)x_i + b(t)u + f(t)\right)\end{aligned} \tag{50}$$

For the simplicity taking $c_n = 1$ and transforming the system (49) in the coordinate of σ, one can write

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= \sigma - \sum_{i=1}^{n-1} c_i x_i \\ \dot{\sigma} &= c_1x_2 + c_2x_3 + \cdots - \sum_{i=1}^{n-1} a_i(t)x_i - a_n\left(\sigma - \sum_{i=1}^{n-1} c_i x_i\right) + b(t)u + f(t)\end{aligned} \tag{51}$$

After choosing the proper value of switching controller u, $\sigma = 0$ and dynamics of $\dot{\sigma}$ is collapsed in finite time.

Hence reduced order dynamics is given as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= -\sum_{i=1}^{n-1} c_i x_i \end{aligned} \tag{52}$$

which is free from the disturbances and system uncertainties.

Thank you for your attention.