Sliding Mode Control: An Introduction

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Outline

What is the ’Sliding mode’ and how did its study start?

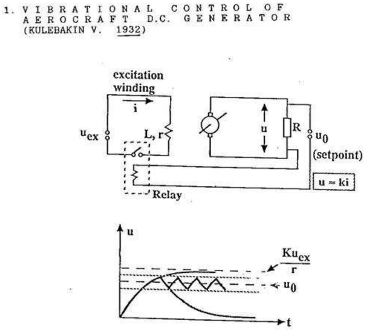


Figure : Primitive Examples - Electrical

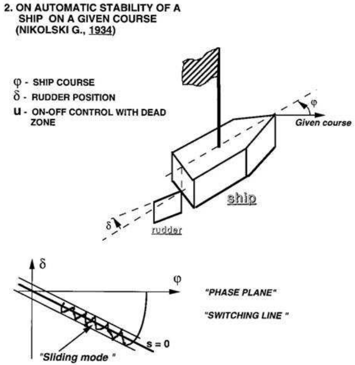


Figure : Primitive Examples-Mechanical

The Sliding Mode Control: Revisited

Variable Structure Control: Prof.Emelyanov

- The first steps of sliding mode control theory originated in the early 1950 initiated by S. V. Emelyanov.
- Started as Variable Structure Control (VSC):-Varying system structure for stabilization.

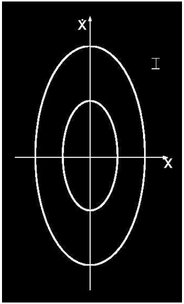


Figure : Mode I $\ddot{x} = -\alpha_1 x$

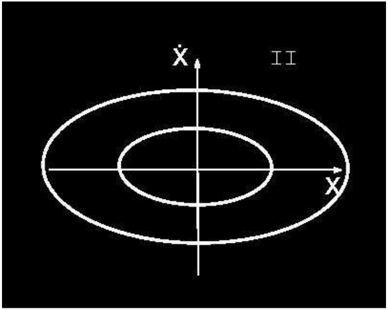
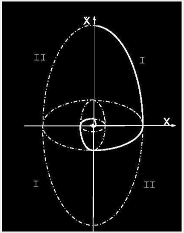


Figure : Mode II $\ddot{x} = -\alpha_2 x$, $0 < \alpha_2 < \alpha_1$

The Sliding Mode Control: Revisited

Piecing together



Properties of VSC

- Both constituent systems were oscillatory and were not asymptotically stable.
- Combined system is asymptotically stable.
- Property not present in any of the constituent system is obtained by VSC.

The Sliding Mode Control: Revisited

Unstable Constituent Systems

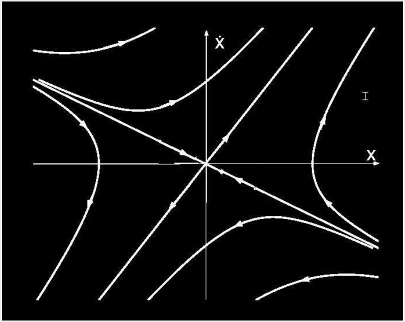


Figure : $\ddot{x} - \xi\dot{x} - \alpha x = 0$

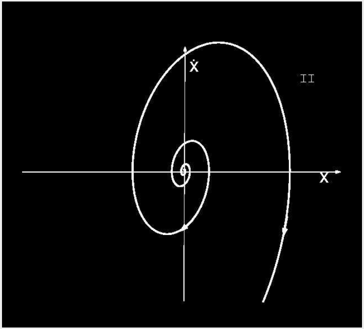


Figure : $\ddot{x} - \xi\dot{x} + \alpha x = 0$

The Sliding Mode Control: Revisited

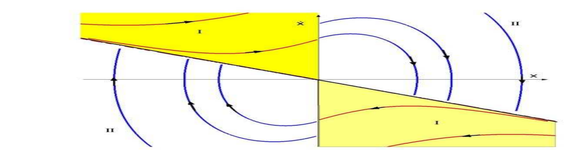
Analysis

- Both systems are unstable
- Only stable mode is one mode of system

$$\ddot{x} - \xi\dot{x} - \alpha x = 0, \lambda = \frac{\xi}{2} - \sqrt{\frac{\xi^2}{4} + \alpha}$$

- If the following VSC is employed

$$Mode = \begin{cases} I & \text{if } xs \leq 0 \\ II & \text{if } xs > 0. \end{cases}, s = cx + \dot{x}, c = -\lambda$$

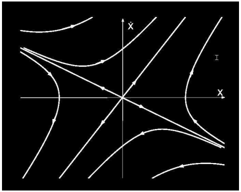


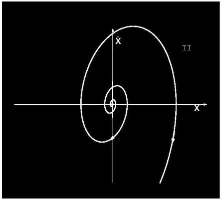
The Sliding Mode Control: Revisited

Unstable Constituent Systems: More Analysis

- Again, property not present in constituent systems is found in the combined system.
- A stable structure can be obtain by varying between two unstable structures.
- However, a more interesting behavior can be observed if we use a different switching logic.

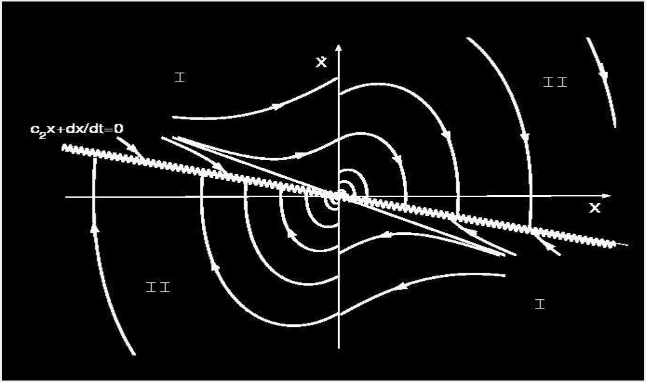
$$Mode = \begin{cases} I & \text{if } xs \leq 0 \\ II & \text{if } xs > 0. \end{cases}, s = \hat{c}x + \dot{x}, 0 < \hat{c} \leq (c = -\lambda)$$





The Sliding Mode Control: Revisited

New trajectory that was not present in any of the two original systems



Sliding Mode:-Motion of the system trajectory along a chosen line/plane/surface of the state space.

Sliding Mode Control Design

The Linear System

$$\dot{x} = Ax + B(u + \rho(t, x))$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}$

- The sliding surface is designed as $\sigma(x) = Cx = 0$.
- The sliding mode control is designed to reach the sliding surface in finite time.

Assumptions

- The pair (A, B), is controllable.
- The scalar $CB \neq 0$.
- The disturbance/uncertainties are matched and bounded, i.e.,$|\rho(t, x)| \leq D_M$.
- The zero dynamics(reduced order dynamics) of the system with the output $\sigma(x)$ is stable.

Sliding Mode Control Design

The Controller Design

Since the function $\sigma(x)$ is to be made zero, it helps to look at its derivative.

$$\dot{\sigma} = C\dot{x} \tag{2.1}$$

$$= CAx + CB(u + \rho(t,x)) \tag{2.2}$$

Now, let us design the control as,

$$u = (CB)^{-1}\left(-CAx + \phi(\sigma)\right)$$

Substituting this control in (2.2) gives,

$$\dot{\sigma} = \phi(\sigma) + CB\rho(t,x) \tag{2.3}$$

- The function $\phi(\sigma)$ must be selected such that, the equation (2.3) provides finite time convergence of σ to zero, in presence of $\rho(t,x)$.
- Once the sliding surface is reached, i.e., $\sigma(x) \equiv 0$, the system trajectory is governed by the zero dynamics, also called reduced order dynamics.

Sliding Mode Control Design

Examples of $\phi(\sigma)$

1. $\phi(\sigma) = -K\ sgn(\sigma)$
2. $\phi(\sigma) = -\lambda\sigma - K\ sgn(\sigma)$
3. $\phi(\sigma) = -K|\sigma|^{\alpha}\ sgn(\sigma)$

However, the third choice can not reject the disturbances if $\rho(t,x) \neq 0$ at $x = 0$.

The Chattering Problem

The control is discontinuous, resulting in high frequency switching by the actuator.

The Relative Degree Problem

$$\dot{\sigma} = CAx + CB(u + \rho(t,x)) \tag{2.4}$$

- If $CB = 0$, then control does not appear in the expression for $\dot{\sigma}$.
- Usually, C is designed surface matrix so $CB = 0$ can be avoided.

However, σ may be some output of the system and C is determined by system dynamics. The objective is to make the output zero robustly, then with $CB = 0$, the sliding mode control design is not possible.

A Brief Review of Filippov Theory

Convexification

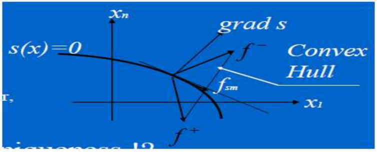


Figure : Filippov Solution

Physical Interpretation of Filippov Method Using Regularization Approach

Consider the system of the following form

$$\dot{x} = f(x, u), \quad x, f \in \mathbb{R}^n, \quad u(x) \in \mathbb{R} \tag{3.1}$$

$$u(x) = \begin{cases} u^+ & \text{if } s(x) > 0 \\ u^- & \text{if } s(x) < 0 \end{cases} \tag{3.2}$$

where the component of vector f, scalar functions u^+, u^- and $s(x)$ are continuous and smooth, and $u^+(x) \neq u^-(x)$.

A Brief Review of Filippov Theory

Physical Interpretation of Filippov Method Using Regularization Approach

- We assume that sliding mode occurs on the sliding surface $s(x) = 0$.
- Try to derive the motion equations using the regularization method.
- Let the discontinuous control be implemented with some imperfections of unexpected nature, control is assumed to take one of the two extreme values, u^+ or u^-, and the discontinuity points are isolated in time.
- Since discontinuity points are isolated in time, the solution exists in the conventional sense beyond the sliding surface.
- Further assume that the state velocity vector $f^+ = f_1 = f(x, u^+)$ and $f^- = f_2 = f(x, u^-)$ to be constant for some point x on the surface $s(x) = 0$ within a short interval $[t, t + \Delta t]$.
- Let the time interval Δt consists of two sets of intervals Δt_1 and Δt_2 such that $\Delta t = \Delta t_1 + \Delta t_2$, where Δt_1 and Δt_2 is the amount of that time when control of magnitude u^+ and u^- is active.

Physical Interpretation of Filippov Method Using Regularization Approach

Mathematically increment of the state vector in this interval Δt is given by

$$\Delta x = f_1 \Delta t_1 + f_2 \Delta t_2 \tag{3.3}$$

and the average state velocity is given as

$$\begin{aligned} \dot{x}_{\text{average}} &= \frac{\Delta x}{\Delta t} = \frac{f_1 \Delta t_1 + f_2 \Delta t_2}{\Delta t_1 + \Delta t_2} = f_1 \frac{\Delta t_1}{\Delta t_1 + \Delta t_2} + f_2 \frac{\Delta t_2}{\Delta t_1 + \Delta t_2} \\ &= \alpha f_1 + (1-\alpha) f_2 \end{aligned} \tag{3.4}$$

where $\alpha = \frac{\Delta t_1}{\Delta t}$ is relative time for control to take value u^+ and $(1-\alpha)$ to take value u^- and also $0 \leq \alpha < 1$.

To get the velocity vector $\dot{x}$ along the sliding surface we have to take limit $\Delta t \to 0$. Hence sliding motion is represented as

$$\dot{x} = \alpha f_1 + (1-\alpha) f_2 \tag{3.5}$$

Physical Interpretation of Filippov Method Using Regularization Approach

Remark:-One can also interpret Eqn.(3.5) as the velocity vector in the vicinity of a point on a discontinuous surface which is complemented by a minimal convex set, and the state velocity vector of the sliding motion belongs to this set.

Because the state trajectories during sliding mode are in the sliding surface $s = 0$, the parameter α should be selected such that the state velocity vector of the system (3.5) is in the tangential plane to the sliding surface.

Mathematically one can write

$$\dot{s} = \nabla[s(x)].\dot{x} = \nabla[s(x)](\alpha f_1 + (1-\alpha)f_2) = 0 \tag{3.6}$$

where $\nabla[s(x)] = \left[\frac{\partial s}{\partial x_1} \cdots \frac{\partial s}{\partial x_n}\right]$. The solution of above equation is given by

$$\alpha = \frac{\nabla(s).f_2}{\nabla(s).(f_2 - f_1)} \tag{3.7}$$

Substituting the α from Eqn.(3.8) to (3.5), one can get motion in sliding mode as

$$\dot{x} = f_{\text{sliding}} = \frac{\nabla(s).f_2}{\nabla(s).(f_2 - f_1)}f_1 - \frac{\nabla(s).f_1}{\nabla(s).(f_2 - f_1)}f_2 \tag{3.8}$$

Equivalent Control Method

Remark:-

Sliding mode occurs in the surface $s(x) = 0$, therefore, the function s and $\dot{s}$ have different signs in the vicinity of the surface and $\dot{s}^{+} = \nabla(s).f_1 < 0$, $\dot{s}^{-} = \nabla(s).f_2 > 0$. Also one can easily check that $\dot{s} = \nabla(s)f_{\text{sliding}} = 0$ for the trajectories of system (3.8) and show that they are confined to the switching surface $s(x) = 0$.

Equivalent Control Method by the Geometrical Point of View

- In sliding mode control, our main aim is to design a control law so that the state trajectories are confined to a sliding manifold in finite time.
- From a geometrical point of view, the equivalent control method does the same job. It replace the discontinuous control on the intersection of the switching surface by a continuous one such that, the state velocity vector lies in the tangential manifold.

Mathematically, consider the system

$$\dot{x} = f(x) + B(x)u \quad x,\ f(x) \in \mathbb{R}^n,\ \ B(x) \in \mathbb{R}, u \in \mathbb{R}^n \tag{3.9}$$

Equivalent Control Method

$u(x)$ is defined as

$$u(x) = \begin{cases} u^{+} & \text{if } s(x) > 0 \\ u^{-} & \text{if } s(x) < 0 \end{cases} \tag{3.10}$$

So,

$$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = G(x)f(x) + G(x)B(x)u_{\text{equivalent}} = 0 \tag{3.11}$$

where $G = \frac{\partial s}{\partial x}$. Assuming the matrix GB is nonsingular for any x, find the equivalent control $u_{\text{equivalent}}$ as the solution of the Eqn.(3.11)

$$u_{\text{equivalent}} = -G(x)B(x)^{-1}G(x)f(x) \tag{3.12}$$

and substituting $u_{\text{equivalent}}$ into (3.9) to yield the sliding mode equation $s = 0$ as

$$\dot{x} = f(x) - G(x)B(x)^{-1}B(x)G(x)f(x) \tag{3.13}$$

Equivalent Control Method

Physical Interpretation of the Equivalent Control

- For the occurrence of the ideal sliding mode it was assumed that the control changes at high (theoretically infinite) frequency such that the state vector is oriented precisely along the intersection of discontinuity surfaces.
- In reality however, various imperfections make the state oscillate in some vicinity of the intersection and control components are switched at finite frequency alternatively taking the positive and negative values.
- These oscillations have high frequency as well as slow components.
- All most all plants under control act as a low pass filter.
- Due to this low pass filter characteristic high frequency component is filtered out, and its motion in sliding mode is determined by the slow component.
- Practically it is reasonable to assume that the equivalent control is close to the slow component of the real control, which can be derived by filtering out the high-frequency components using a low-pass filter.

Physical Interpretation of the Equivalent Control

Mathematically, the output of a low-pass filter

$$\tau \dot{z} + z = u \quad (4.1)$$

tends to the equivalent control

$$\lim_{\tau \to 0, \frac{\Delta}{\tau} \to 0} z = u_{\text{equivalent}} \quad (4.2)$$

where τ, Δ are the time constant of low pass filter and width of the manifold respectively.

Remark:-

- To eliminate the high-frequency component of the control in sliding mode, the frequency should be much higher than $\frac{1}{\tau}$, or $\frac{1}{f} << \tau$, hence, $\Delta << \tau$.
- Finally, the time constant of the low-pass filter should be made to tend to zero because the filter should not distort the slow component of the control.

Invariance in Sliding-modes

B. Drazenovic “The invariance conditions in variable structure systems” Automatica, v.5, No.3, Pergamon Press, 1969.

Figure : Prof.B. Drazenovic

Mathematical Treatment of Matched Uncertainty

Mathematical meaning of equivalent control is that, when the trajectories are on the sliding surface then σ and $\dot{\sigma}$, both must be zero. But this assumption is valid only when the control input appears linearly for a given state. Now consider the following system

$$\dot{x} = f(x) + g(x)u + d \tag{5.1}$$

Invariance in Sliding-modes

Consequently, one can write

$$\dot{\sigma} = \frac{\partial\sigma}{\partial x}\dot{x} = \frac{\partial\sigma}{\partial x}f(x) + \frac{\partial\sigma}{\partial x}g(x)u + \frac{\partial\sigma}{\partial x}d \tag{5.2}$$

For calculating equivalent value of control substitute $\dot{\sigma} = 0$ in Eqn.(5.2), the only condition is that $\frac{\partial\sigma}{\partial x}g(x) \neq 0$, we can write

$$u_{\text{equivalent}} = -\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\left(\frac{\partial\sigma}{\partial x}f(x) + \frac{\partial\sigma}{\partial x}d\right) \tag{5.3}$$

The closed loop dynamics during sliding is obtained by substituting (5.3) into (5.1), one can write

$$\begin{aligned}\dot{x} &= f(x) - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\left(\frac{\partial\sigma}{\partial x}f(x) + \frac{\partial\sigma}{\partial x}d\right) + d \\ &= \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)f(x) + \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)d \end{aligned} \tag{5.4}$$

Invariance in Sliding-modes

Now suppose that disturbance d is entering through control channel. Therefore one can write $d = g(x)\delta(t)$, where $\delta(t)$ is unknown signal, but with a known bound. This kind of disturbances are known as matched disturbances.

After substituting the value of d, into (5.4), one can write

$$\begin{aligned}
\dot{x} &= \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)f(x) + \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)g(x)\delta(t) \\
&= \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)f(x) + g(x)\delta(t) - g(x)\delta(t) \\
&= \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)f(x) \qquad (5.5)
\end{aligned}$$

Hence the closed loop dynamics which is given in (5.5), is completely independent of any matched disturbances, when the system is in the sliding mode. This property is known as the **invariance property** with respect to disturbance.

Sliding Mode Control Design Using Regular Form Transformation

Consider the linear time invariant system of the following form

$$\dot{x}(t) = (A + \Delta A(t))x(t) + Bu + Df(t) \tag{6.1}$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{m\times 1}$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the state vector, control input, system matrix and input matrix respectively. $\Delta A(t)$ and $f(t)$ represent parameter variations and disturbance vector, where D is the disturbance input matrix.

Assumptions:-

- The system is assumed to be controllable i.e the controllability matrix $[B \;\; AB \;\; A^2B \cdots A^{n-1}B]$ has full rank.
- $\Delta A \in \text{range}(B)$ and $D \in \text{range}(B)$, or there exist constant or time varying matrices Ψ_1 and Ψ_2, such that

$$\Delta A = B\Psi_1, \;\; D = B\Psi_2 \tag{6.2}$$

Using Eqn.(6.1) and assumption (6.2), one can write

$$\dot{x}(t) = Ax(t) + B(u + \underbrace{\Psi_1 x(t) + \Psi_2 f(t)}) = Ax(t) + B(u + \Psi_0), \Psi_0 = \Psi_1 x(t) + \Psi_2 f(t) \tag{6.3}$$

Sliding Mode Control Design Using Regular Form Transformation

Regular Form Transformation:

Input matrix B in Eqn.(6.1) may be partitioned (after reordering the state vector components)as

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{6.4}$$

where $B_1 \in \mathbb{R}^{(n-m)\times m}$, $B_2 \in \mathbb{R}^{m\times m}$ with $\det B_2 \neq 0$.

The nonsingular coordinate transformation T, which is able to convert system (6.1) into regular form is given by

$$T = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix} \tag{6.5}$$

and change of coordinate is

$$Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{6.6}$$

Sliding Mode Control Design Using Regular Form Transformation

Transformed system (6.1), after applying above change of coordinate

$$\begin{aligned}\dot{z}_1(t) &= A_{11}z_1(t) + A_{12}z_2(t)\\ \dot{z}_2(t) &= A_{21}z_1(t) + A_{22}z_2(t) + u + \Psi_0\end{aligned} \tag{6.7}$$

where $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^{m}$ and

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{6.8}$$

Design of Sliding Surface:-

Now our next aim is to design a specified switching manifold such that after reaching the manifold, the system is free from its dynamics. The designed manifold can be linear or nonlinear, depends on our specified goal. The simplest manifold is a linear one. So, in this chapter we focus towards linear manifolds only, unless otherwise specified.

Sliding Mode Control Design Using Regular Form Transformation

The linear switching manifold (sliding surface) is given as follow

$$S = C_1 z_1 + z_2 \tag{6.9}$$

where $S \in \mathbb{R}^{m\times 1}$ and $C_1 \in \mathbb{R}^{m\times(n-m)} > 0$.

Now taking the first time derivative of the sliding surface (6.9) and substituting the value from the Eqn.(6.7), one can get

$$\begin{aligned} \dot{S} &= C_1\dot{z}_1 + \dot{z}_2 \\ &= (C_1A_{11} + A_{21})z_1(t) + (C_1A_{12} + A_{22})z_2(t) + u + \Psi_0 \end{aligned} \tag{6.10}$$

Now expressing the system (6.7) into z_1 and S coordinate, using the differential Eqn.(6.9) and (6.10)

$$\begin{aligned} \dot{z}_1(t) &= A_{11}z_1(t) + A_{12}(S - C_1z_1(t)) \\ \dot{S} &= (C_1A_{11} + A_{21})z_1(t) + (C_1A_{12} + A_{22})(S - C_1z_1(t)) + u + \Psi_0 \\ &= (C_1A_{11} + A_{21} - C_1A_{12}C_1 - A_{22}C_1)z_1(t) + (C_1A_{12} + A_{22})S + u + \Psi_0 \end{aligned} \tag{6.11}$$

Sliding Mode Control Design Using Regular Form Transformation

Existence Condition of Sliding Mode:-The main objective here is to design a control u such that, the sliding motion occurs in finite time. For this, the control is selected according to the following theorem

Theorem:-

The control input u which is defined as

$$u = \nu - (C_1A_{11} + A_{21} - C_1A_{12}C_1 - A_{22}C_1)z_1(t) - (C_1A_{12} + A_{22})S \quad (6.12)$$

where,

$$\nu = -K_1S - K_2\text{sign}(S)) \quad (6.13)$$

with $K_1 > 0, K_2 > \|\Psi_0\|$ leads to the establishment S equal to zero in finite time.

Proof:-

Substituting u from Eqn.(6.12) to (6.11), one can write

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}(S - C_1z_1(t))$$
$$\dot{S} = -K_1S - K_2\text{sign}(S)) + \Psi_0 \quad (6.14)$$

Sliding Mode Control Design Using Regular Form Transformation

Choosing the Lyapunov function as $V = \frac{1}{2}S^T S$, and calculating the time derivative of the Lyapunov function along (6.14), one can get

$$\begin{aligned} \dot{V} &= S^T \dot{S} = S^T(-K_1 S - K_2 \text{sign}(S)) + \Psi_0) \\ &= -K_1 V - K_2\|S\| + S^T \Psi_0 \end{aligned} \tag{6.15}$$

Using Cauchy-Schwartz inequality $S^T \Psi_0 \leq \|S\|\|\Psi\|$ and $(2V)^{\frac{1}{2}} = (S^T S)^{\frac{1}{2}} = \|S\|$, we get

$$\begin{aligned} \dot{V} &\leq -K_1 V - (K_2 - \|\Psi\|)\|S\| \\ &\leq -K_1 V - 2^{\frac{1}{2}}(K_2 - \|\Psi\|)V^{\frac{1}{2}} \\ &\leq -K_1 V - \eta V^{\frac{1}{2}} \leq 0 \end{aligned} \tag{6.16}$$

Therefore, one can ensure the asymptotic stability of the trajectory to the sliding manifold where $\eta = 2^{\frac{1}{2}}(K_2 - \|\Psi\|)$.

Sliding Mode Control Design Using Regular Form Transformation

Now, we have to prove that convergence takes place in finite time.

Time calculation:-

$$\begin{aligned}
\dot{V} &= \frac{dV}{dt} \leq -k_1 V - \eta V^{\frac{1}{2}} \\
T &\leq \int_{V(0)}^{0} \frac{dV}{-k_1 V - \eta V^{\frac{1}{2}}} = \int_{V(0)}^{0} \frac{dV}{-V^{\frac{1}{2}}\left(k_1 V^{\frac{1}{2}} + \eta\right)} \\
&= \frac{2}{k_1}\left(ln\left(k_1 V(0)^{\frac{1}{2}} + \eta\right) - ln(\eta)\right) \qquad (6.17)
\end{aligned}$$

Therefore, the above claim is justified since the time T is always finite.

Equivalent Control

It is clear from Eqn.(6.12), during the sliding mode $S = 0$ and average value of $\nu = 0$, therefore the remaining control is only $u_{\text{equivalent}} = -(C_1 A_{11} + A_{21} - C_1 A_{12} C_1 - A_{22} C_1) z_1(t)$, which is theoretically interpreted as **equivalent control**, which is the control required to maintain a sliding motion on the manifold S.

Sliding Mode Control Design Using Regular Form Transformation

Analysis of Reduced Order Dynamics

Therefore, after time T, dynamics of S has been collapsed and the reduced order dynamics of system (6.14) after substituting $S = 0$, is given as

$$\dot{z}_1(t) = (A_{11} - A_{12}C_1)z_1(t) \tag{6.18}$$

The stability and performance of the above reduced order system depends on (A_{11}, A_{12}). Therefore, the design of C_1 depends on the controllability of the pair (A_{11}, A_{12}).

Lemma

The matrix pair (A_{11}, A_{12}) is controllable, if and only if the pair (A, B) is controllable.

Proof:-

$$\begin{aligned}\text{rank}[sI - A \;\; B] &= \text{rank}\begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ -A_{21} & sI - A_{22} & I \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} sI - A_{11} & A_{12} \end{bmatrix} + m \;\; \text{for all } s \in \mathbb{C}\end{aligned}$$

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This implies

$$\operatorname{rank}[sI - A \;\; B] = n \Leftrightarrow \operatorname{rank}\,[sI - A_{11} \;\; A_{12}] = n - m$$

and from the Popov-Belevitch-Hautus (PBH) rank test, it follows that (A, B) is controllable if and only if the pair (A_{11}, A_{12}) is controllable.□

Now one can design C_1 using any robust linear state feedback method, such as quadratic minimization, direct or robust eigenvalue assignment, using LMI etc.

Thank You!