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$$\mathbb{P}\{|x(t)| < \infty,\ \forall t \in \mathbb{R}_+\} = 1, \quad \forall x(0) \in \mathbb{R}^N \tag{58}$$

$$\mathbb{P}\left\{ \underline{\alpha}(|x(t)|) < \frac{1}{\epsilon}\left(\overline{\alpha}(|x(0)|) + \int_0^l \sigma(|r(\tau)|)\,d\tau\right), \right.$$
$$\left. \forall t \in [0,l] \right\} \geq 1-\epsilon,\ \forall l \in \mathbb{R}_+, \forall x(0) \in \mathbb{R}^N \setminus \{0\},\ \forall \epsilon \in (0,1). \tag{59}$$

⋮

$$\mathbb{E}[V(x(t_A \wedge t))] \leq V(x(0)) + e(t) \tag{62}$$

follows from $\mathcal{L}V \leq \sigma(|r|)$. Using $\mathbb{P}\{t_A \leq t\} \inf_{|y|\geq A} V(y) \leq \mathbb{E}[V(x(t_A \wedge t))]$ implied by (61), from (62) we obtain

$$\mathbb{P}\{t_A \leq t\} \leq \frac{V(x(0)) + e(t)}{\underline{\alpha}(A)}. \tag{63}$$

⋮

(5). By definition, we have $v(0) = V(x(0)) \leq z(0)$ and $v(t) \geq 0$ for all $t \in \mathbb{R}_+$. Given $l \in \mathbb{R}_+$, for each ϵ, $x(0)$ and r, define $T(l) \in [0,\infty]$ as

$$T(l) := \inf\{t \geq 0 : v(t) \geq z(l)\}, \tag{65}$$

⋮

given ϵ, $x(0)$ and r it holds for each $l \in \mathbb{R}_+$ that

$$\{T(l) \leq t\} \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_+. \tag{66}$$

Thus, applying the argument of [17, Proof of Lemma 3.2, p.73] to the stopped process $x(T \wedge t)$ with (66), we obtain

$$\mathbb{E}[v(T \wedge t)] = V(x(0)) + \mathbb{E}\left[\int_0^{T\wedge t} \mathcal{L}V(x(\tau))d\tau\right]$$

for each $t \in \mathbb{R}_+$. Property $\mathcal{L}V \leq \sigma(|r|)$ yields

$$\mathbb{E}[v(T \wedge t)] \leq V(x(0)) + e(t) \tag{67}$$

since $T \wedge t \leq t$. The definition of T and $v(t) \geq 0$ yield

$$\mathbb{E}[v(T \wedge t)] \geq \mathbb{E}[I_{\{T\leq t\}} v(T)] = z(l)\mathbb{P}\{T \leq t\}, \tag{68}$$

where $I_{\{T\leq t\}}$ is the indicator function of the set $\{T \in \mathbb{R}_+ : T \leq t\}$. Combining (68) with (67) yields

$$V(x(0)) + e(t) \geq z(l)\mathbb{P}\{T(l) \leq t\} \tag{69}$$

for each $l \in \mathbb{R}_+$. Substituting (64) into (69) gives

$$\epsilon \geq \mathbb{P}\{T(l) \leq t\}, \quad \forall t \in [0,l]. \tag{70}$$

By virtue of T defined in (65) with (60) and (64) and the property $\underline{\alpha}(|x(t)|) \leq V(x(t)) = v(t)$, using (70), we arrive at (59). Q.E.D.

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0. Applying (73) to this property yields

$$\mathbb{E}[W(t_A \wedge t)] \leq W(0) - \mathbb{E}\left[\int_0^{t_A\wedge t} \alpha(W(\tau))d\tau\right] + \int_0^t \bar{\sigma}(|r(\tau)|)d\tau,$$

where $\alpha \in \mathcal{K}$. The remainder of the proof proceeds in the same way as the proof of Theorem 2 with Lemma 1 and $A \to \infty$.

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The components w_i of $w \in \mathbb{R}^S$ are again mutually independent standard Wiener processes. The (k,l)-component of Θ represents the intensity describing the influence of the l-th component of $w(t)$ on $x(t)$ through the k-th column of $h(x)$. In fact, the deterministic function $\Theta(t)\Theta(t)^T$ is the infinitesimal variance matrix of the S-dimensional stochastic process represented by $\Theta(t)dw$ in (10). We assume $f(0)=0$. It is stressed that for (10), we do not assume $h(0)=0$. This paper employs the notion of noise-to-state stability for system (10) introduced in [18].

~~1)~~ *Definition 5:* System (10) is said to be noise-to-state stable (NSS) if for each $\epsilon \in (0,1)$, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that

$$\mathbb{P}\left\{|x(t)| < \beta(|x(0)|,t) + \gamma\left(\sup_{\tau\in[0,t]}\left|\Theta(\tau)\Theta^T(\tau)\right|_{\mathbf{F}}\right)\right\} \geq 1-\epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0)\in\mathbb{R}^N\setminus\{0\}. \tag{11}$$

NSS defines robustness with respect to the noise variance $\Theta(t)\Theta(t)^T$. This idea contrasts with the one employed by another type of ISS proposed and investigated in [28], [31] where r in (5) is a random variable in addition to the Wiener process w. As we did for (5), we define the following two properties for system (10):

~~2)~~ *Definition 6:* System (10) is said to be integral noise-to-state stable (iNSS) if for each $\epsilon \in (0,1)$, there exists a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function μ and a class $\mathcal{K}_\infty$ function χ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \beta(|x(0)|,t) + \int_0^t \mu\left(\left|\Theta(\tau)\Theta^T(\tau)\right|_{\mathbf{F}}\right)d\tau\right\} \geq 1-\epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0)\in\mathbb{R}^N\setminus\{0\}. \tag{12}$$

~~3)~~ *Definition 7:* System (10) is said to be quasi-integral noise-to-state stable (quasi-iNSS) if there exists a constant $R > 0$ satisfying the following: for each $\epsilon \in (0,1)$, there exist a class $\mathcal{KL}$ function β, class $\mathcal{K}$ functions μ, γ, and class $\mathcal{K}_\infty$ functions χ, $\overline{\beta}$ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \overline{\beta}(|x(0)|) + \int_0^t \mu\left(\left|\Theta(\tau)\Theta^T(\tau)\right|_{\mathbf{F}}\right)d\tau\right\} \geq 1-\epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0)\in\mathbb{R}^N\setminus\{0\} \tag{13}$$

$$\sup_{\tau\in[0,\infty)}\left|\Theta(\tau)\Theta^T(\tau)\right|_{\mathbf{F}} < R \Rightarrow (11). \tag{14}$$

In Definitions 5–7, we do not require the influence of the

IV. LYAPUNOV CHARACTERIZATIONS

For any given $\mathbf{C}^2$ function $V: x\in\mathbb{R}^N \mapsto V(x)\in\mathbb{R}_+$, the infinitesimal generator $\mathcal{L}$ associated with systems (3), (5) and (10) is defined as

$$\mathcal{L}V = \frac{\partial V}{\partial x}f + \frac{1}{2}\mathrm{Tr}\left\{Q^T h^T \frac{\partial^2 V}{\partial x^2} hQ\right\} \tag{15}$$

where

$$Q = I \quad \text{for (3) and (5)}$$
$$Q = \Theta(t) \quad \text{for (10).} \tag{16}$$

Here, the symbol I denotes the identity matrix of size $S\times S$.

A. Robustness With Respect to Deterministic Disturbance

For ISS, the following characterization is available, which is parallel to the deterministic case [27].

Proposition 1: Consider (5). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V:\mathbb{R}^N\to\mathbb{R}_+$, and continuous functions $\rho\in\mathcal{K}$ and $\eta\in\mathcal{P}$ such that the implication

$$V(x) \geq \rho(|r|) \Rightarrow \mathcal{L}V \leq -\eta(V(x)) \tag{17}$$

holds for all $x\in\mathbb{R}^N$ and $r\in\mathbb{R}^M$, then system (5) is ISS in probability.

As indicated in [29], the proof of Proposition 1 essentially follows an adaptation of the one given in [18] which is demonstrated in detail in [20]. Note that applying [17, Theorem 5.1] or [21, Theorem 2.4 in Section 4.2] to the proof of [18, Theorem 3.3] allows us to replace $\eta\in\mathcal{K}$ with $\eta\in\mathcal{P}$. A related discussion on Proposition 1 is given in Appendix H. The main developments in this subsection are the following two theorems establishing quasi-iISS and iISS in probability.

Theorem 1: Consider (5). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V:\mathbb{R}^N\to\mathbb{R}_+$, and continuous functions $\alpha\in\mathcal{K}$ and $\sigma\in\mathcal{K}$ such that

$$\mathcal{L}V \leq -\alpha(V(x)) + \sigma(|r|) \tag{18}$$

holds for all $x\in\mathbb{R}^N$ and $r\in\mathbb{R}^M$, then system (5) is quasi-iISS in probability.

In [20], [33], [34], the function α in (18) is assumed to be of class $\mathcal{K}_\infty$ in order to obtain ISS of (5). Indeed, if $\alpha\in\mathcal{K}_\infty$ holds, inequality (17) is satisfied for any $\tau>1$ with $\rho=\alpha^{-1}\circ\tau\sigma\in\mathcal{K}$ and $\eta=(1-1/\tau)\alpha\in\mathcal{K}$. As in the deterministic case, we can relax $\alpha\in\mathcal{K}_\infty$ into

$$\lim_{s\to\infty}\alpha(s) \geq \lim_{s\to\infty}\sigma(s) \tag{19}$$

in establishing ISS of (5) from (18). This fact can be verified by the choice $\rho=\alpha^{-1}\circ(\mathbf{Id}+\omega)\circ\sigma\in\mathcal{K}$ yielding (17) with

Therefore, the function V satisfying (18) establishes not only quasi-iISS but also ISS of the stochastic system (5) in both cases of $\alpha\in\mathcal{K}_\infty$ and (19). In the deterministic case, the function $\rho=\alpha^{-1}\circ\tau\sigma$ (resp. $\rho=\alpha^{-1}\circ(\mathbf{Id}+\omega)\circ\sigma$) for a constant $\tau>1$ (resp. a continuous function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

tinuous functions $\rho\in\mathcal{K}$ and $\eta\in\mathcal{P}$ such that the implication

$$V(x) \geq \rho\left(|\Theta\Theta^T|_{\mathbf{F}}\right) \Rightarrow \mathcal{L}V \leq -\eta(V(x)) \tag{24}$$

holds for all $x\in\mathbb{R}^N$, then system (10) is NSS.

A function V satisfying the conditions of Proposition 2 is

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generator of the transformed (scaled, filtered) Lyapunov function $\hat{V}_i$ associated with the x_i-subsystem is computed as

$$\mathcal{L}\hat{V}_i \leq \lambda_i\left(F_i^{-1}\left(\hat{V}_i(x_i)\right)\right)\left\{-\alpha_i\left(F_i^{-1}\left(\hat{V}_i(x_i)\right)\right) + \cancel{\sigma_i}\ \sigma_i\left(F_{3-i}^{-1}\left(\hat{V}_{3-i}(x_{3-i})\right)\right) + \kappa_i(|r_i|)\right\} + \frac{1}{2}\lambda_i'(V_i(x_i))\,\mathrm{Tr}\left\{h_i^T\left(\frac{\partial V_i}{\partial x_i}\right)^T\left(\frac{\partial V_i}{\partial x_i}\right)h_i\right\} \tag{30}$$

from (15) with $Q=I$

$$\mathrm{Tr}\left\{h_i^T\frac{\partial^2\hat{V}_i}{\partial x_i^2}h_i\right\} = \lambda_i(V_i(x_i))\,\mathrm{Tr}\left\{h_i^T\frac{\partial^2 V_i}{\partial x_i^2}h_i\right\} + \lambda_i'(V_i(x_i))\,\mathrm{Tr}\left\{h_i^T\left(\frac{\partial V_i}{\partial x_i}\right)^T\left(\frac{\partial V_i}{\partial x_i}\right)h_i\right\}$$

[5] In this paper, "with respect to the input x_{3-i}" means that the remaining input r_i is supposed to be zero. In addition, when we refer to a stability property of an individual x_i-subsystem, the x_i-subsystem is disconnected from

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T_i in (30) for establishing stability of interconnected systems.

Theorem 5: Consider (5) consisting of (26) and (27). Suppose that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2\in\mathcal{K}$ and $c>2$ such that

$$\tilde{\alpha}_i(s) \leq \alpha_i(s) - \frac{1}{2}\frac{\sigma_{3-i}'(s)}{\sigma_{3-i}(s)}T_i(s), \quad \forall s\in\mathbb{R}_+,\ i=1,2 \tag{33}$$

$$\tilde{\alpha}_1^{\ominus}\circ c\sigma_1\circ\tilde{\alpha}_2^{\ominus}\circ c\sigma_2(s) \leq s, \quad \forall s\in\mathbb{R}_+ \tag{34}$$

hold. Then interconnection (26), (27) is GAS in probability for $r=0$. Moreover, if

$$\left\{\lim_{s\to\infty}\tilde{\alpha}_i(s)=\infty \text{ or } \lim_{s\to\infty}\sigma_{3-i}(s)\kappa_i(1)<\infty\right\},\ i=1,2 \tag{35}$$

is satisfied, the following hold true:

(i) Interconnection (26), (27) is quasi-iISS in probability.
(ii) If there exists $D>0$ such that

$$\left(\frac{\partial V_i}{\partial x_i}(x_i)\right)h_i(x)=0, \quad \forall |x|\geq D,\ i=1,2 \tag{36}$$

[6] If $h_i(x)$ is bounded in x_{3-i}, $T_i(s)<\infty$ is guaranteed for all $s\in\mathbb{R}_+$. In

holds, interconnection (26), (27) is iISS in probability.
(iii) If there exist $D_i>0$, $i=1,2$, such that

$$\left(\frac{\partial V_i}{\partial x_i}(x_i)\right)h_i(x)=0, \quad \forall x\in\left\{x\in\mathbb{R}^N : V_i(x_i)\geq D_i\right\} \tag{37}$$

$$D_i < \lim_{s\to\infty}\sigma_{3-i}^{\ominus}\circ\alpha_{3-i}(s) \tag{38}$$

holds, interconnection (26), (27) is iISS in probability.
(iv) If $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are of class $\mathcal{K}_\infty$, interconnection (26), (27) is ISS in probability.

For notational simplicity, the above theorem employed the

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$$\left\{\lim_{s\to\infty}\alpha_i(s)=\infty \text{ or } \lim_{s\to\infty}\sigma_{3-i}(s)\kappa_i(1)<\infty\right\},\ i=1,2 \tag{41}$$

is satisfied, the following hold true:

(i) Interconnection (26), (27) is quasi-iISS in probability.
(ii) If there exists $D>0$ such that (36) holds, interconnection (26), (27) is iISS in probability.
(iii) If there exist $D_i>0$, $i=1,2$, such that (37) and (38) hold, interconnection (26), (27) is iISS in probability.
(iv) If α_1 and α_2 are of class $\mathcal{K}_\infty$, interconnection (26), (27) is ISS in probability.

If $\mathcal{L}V_1$ and $\mathcal{L}V_2$ are bounded from above by functions matching each other, we can get rid of (33) in Theorem 5, and $c>2$ in (34) can be relaxed into $c>1$ as stated below.

Corollary 2: Consider (5) consisting of (26) and (27). Sup-

Then (i), (ii), (iii), and (iv) in Corollary 1 hold true.

At the price of the matching nonlinearity condition (42) that is quite restrictive for nonlinear systems, the proof of Corollary 2 becomes considerably simpler than that of Theorem 5. In fact, the matching nonlinearity assumption allows us to use constant λ_1 and λ_2 in (29), i.e., linear F_1 and F_2. This idea employed by Corollary 2 has been used as a popular quick recipe in the literature for tackling interconnections of stochastic systems (e.g. [34] and [35]).[7] The use of a constant λ_i which amounts to a linear transformation F_i simply allows us to avoid the stochastic degradation in (30). For deterministic systems, getting rid of the matching nonlinearity

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conditions (35) and (40) in Theorem 5 and Corollary 1 allow us to get rid of the above two deficiencies in [33], and precisely establish ISS described in Definition 2.

B. Robustness With Respect to Stochastic Disturbance

This subsection deals with system (10) consists of

$$dx_1 = f_1(x_1,x_2)dt + h_1(x)\Theta_1(t)dw_1 \tag{44}$$

$$dx_2 = f_2(x_1,x_2)dt + h_2(x)\Theta_2(t)dw_2 \tag{45}$$

where $w_i(t)$ is the S_i-dimensional vector of mutually independent standard Wiener processes for each $i=1,2$. As in (10), we assume $f_i(0,0)=0$, and the (k,l)-component of the matrix $\Theta_i(t)\in\mathbb{R}^{S_i\times S_i}$ denotes the intensity describing the influence

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of the l-th component of $w_i(t)$ on $x_i(t)$ through the k-th column of $h_i(x)$. The matrix $\Theta(t) \in \mathbb{R}^{S\times S}$ is obtained as

$$\Theta(t) = \begin{bmatrix} \Theta_1(t) & 0 \\ 0 & \Theta_2(t) \end{bmatrix}.$$

It is stressed that in contrast to (26) and (27), we do not assume $h_i(0) = 0$ for (44) and (45). The following is assumed throughout this subsection.

Assumption 2: For each $i = 1, 2$, there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, a $\mathbf{C}^1$ function $\alpha_i, \sigma_i \in \mathcal{K}$ and a $\mathbf{C}^0$ function $\omega_i \in \mathcal{K} \cup \{0\}$ such that

$$\mathcal{L}V_i \le -\alpha_i\left(V_i(x_i)\right) + \sigma_i\left(V_{3-i}(x_{3-i})\right) + \omega_i\left(|\Theta_i\Theta_i^T|_{\mathbf{F}}\right) \tag{46}$$

holds for all $x_i \in \mathbb{R}^{N_i}$, $x_{3-i} \in \mathbb{R}^{N_{3-i}}$ and all $\Theta_i \in \mathbb{R}^{S_i\times S_i}$, where ω_i is the zero function, i.e., $\omega_i = 0$ if $h_i = 0$. Here,

is satisfied, then the following hold true:

(i) If

$$\limsup_{s\to\infty} \frac{\alpha_i'(s)}{\alpha_i(s)}\overline{H}_i(s) < \infty, \ i = 1, 2 \tag{51}$$

$$\limsup_{s\to\infty} \frac{\sigma_{3-i}'(s)}{\sigma_{3-i}(s)}\overline{H}_i(s) < \infty, \ i = 1, 2 \tag{52}$$

hold, then interconnection (44), (45) is quasi-iNSS.

(ii) If there exists $D > 0$ such that (36) is satisfied, interconnection (44), (45) is iNSS.

(iii) If there exist $D_i > 0$, $i = 1, 2$, such that (37) and (38) are satisfied, interconnection (44), (45) is iNSS.

(iv) If α_1 and α_2 are of class $\mathcal{K}_\infty$ and (51) and (52) hold, interconnection (44), (45) is NSS.

The difference from quasi-iNSS, iNSS and NSS appears in

From (18) and property (23) in (73) it also follows that:

$$W(t) \ge D \Rightarrow \frac{d}{dt}W(t) \le -\alpha\left(W(t)\right) + \sigma\left(|r(t)|\right). \tag{75}$$

$$\phi\left(h(m)s\right) \le \alpha(s), \quad \forall\, s \in [0, m] \tag{80}$$

for all $m \in \mathbb{R}_+$. Let $m(T) = \overline{D}(V(x(0)), T)$ for all $T \in \mathbb{R}_+$. With the help of (78) and (80), applying Jensen's inequality to

Proposition 1 with $r = 0$ establishes GAS from (92). Next, assume that (35) holds in addition to $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}$ and $c > 2$ satisfying (33) and (34). We again use (89) for V in (29).

(i) Let $\tau \in (2, c)$. The technique in (90) allows one to prove

Thus, Theorems 1 completes the proof.

(ii) Define $\hat{D} = \overline{\alpha}(D)$. Then from $|x| \ge \overline{\alpha}^{-1}(V(x))$ it follows that $V(x) \ge \hat{D}$ implies $|x| \ge D$. Thus, under the assumption (36), replacing D with $\hat{D}$ in Theorem 2 proves the claim with (18) for (97) and (98).

(iii) Suppose that there exist $D_1, D_2 > 0$ satisfying (37) and (38) for $i = 1, 2$. Since (33) means $\tilde{\alpha}(s) \le \alpha(s)$ for $s \in \mathbb{R}_+$, property (34) implies $\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \le s$ for all $s \in \mathbb{R}_+$. Thus, the condition (38) guarantees the existence of $\hat{D}_i \in [D_i, \infty)$, $i = 1, 2$, and $p > 1$ such that

$$-\alpha_i(\hat{D}_i) + p\sigma_i(\hat{D}_{3-i}) \le 0, \quad i = 1, 2. \tag{99}$$

Let $W(t) = V(x(t))$, $Y_i(t) = V_i(x_i(t))$, $\tilde{D}_i = F_i(\hat{D}_i)$ and $W_i(t) = F_i(V_i(x_i(t)))$, where $F_i(s) = \int_0^s \lambda_i(\tau)d\tau$. From (18) and (37) it follows that:

$$W_i(t) \ge \tilde{D}_i, i = 1, 2 \Rightarrow \frac{d}{dt}W(t) \le -\alpha\left(W(t)\right) + \sigma\left(|r(t)|\right) \tag{100}$$

for $\alpha \in \mathcal{K}$ and $\sigma \in \mathcal{K} \cup \{0\}$ given in (97) and (98). Let $\mathbf{B} = \{s \in \mathbb{R}_+^2 : s_i < \tilde{D}_i, i = 1, 2\}$ and $\mathbf{B}^c = \mathbb{R}_+^2 \setminus \mathbf{B}$. Define a sequences of times $\{\tilde{t}_j\}_{j\ge 0}$ as done in the proof of Theorem 2. By definition

$(p/(p-1))\kappa_i]\kappa_i \in \mathcal{K} \cup \{0\}$ and obtain

$$\mathcal{L}W_i \le \bar{\sigma}_i\left(|r_i(t)|\right) \tag{103}$$

by dividing the evaluation of the above $\mathcal{L}W_i$ into the two cases, $\alpha_i(Y_i(t)) \ge (p/(p-1))\kappa_i(|r_i(t)|)$ and $\alpha_i(Y_i(t)) < (p/(p-1))\kappa_i(|r_i(t)|)$. Let $\bar{\sigma}, Z \in \mathcal{K} \cup \{0\}$ and Γ be

$$\bar{\sigma}(s) \ge \max\left\{\bar{\sigma}_1(s) + \bar{\sigma}_2(s), \sigma(s)\right\}, \quad \forall\, s \in \mathbb{R}_+$$

$$\Gamma(s) = s + \tilde{D}_1 + \tilde{D}_2, \ Z(t) = \int_0^t \bar{\sigma}\left(|r(\tau)|\right) d\tau.$$

and define $\overline{D} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\overline{D}(s, t) = \Gamma(s) + Z(t)$ which is continuous and non-decreasing in both $s \in \mathbb{R}+$ and $t \in \mathbb{R}+$. By virtue of the continuity of trajectories, combining (101), (102), and (103) yields

$$\mathbb{P}\left[V\left(x(t)\right) \le \overline{D}\left(V\left(x(0)\right), t\right)\right] = 1, \quad \forall\, t \in \mathbb{R}_+. \tag{104}$$

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Recall that $\mathcal{L}V \le -\alpha(V) + \sigma(|r|)$. Let $t_A \in \mathbb{R}_+$ be the first exit time defined as (61) for an arbitrarily given $A > 0$. Applying (73) and Tonelli's Theorem to this property yields

$$\mathbb{E}\left[W(t_A \wedge t)\right] \le W(0) - \int_0^{t_A\wedge t} \mathbb{E}\left[\alpha\left(W(\tau)\right)\right] d\tau + \int_0^{t_A\wedge t} \bar{\sigma}\left(|r(\tau)|\right) d\tau$$

where $\alpha \in \mathcal{K}$. The remainder of the proof proceeds in the same way as the proof of Theorem 2 with Lemma 1.

(iv) In the case of $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$, we have $\alpha \in \mathcal{K}_\infty$ in (97). Pick $\tau > 1$ and define $\rho = \alpha^{-1} \circ \tau\sigma \in \mathcal{K}$. By virtue of (17) with $\eta = (1 - 1/\tau)\alpha$, Proposition 1 establishes the claim.

G. *Proof of Theorem 6*

Pick $\tau > 0$ and $\varphi \ge 0$ such that

$$1 < \tau < c, \quad \left(\frac{\tau}{c}\right)^{\varphi} \le \tau - 1. \tag{106}$$

Define $V : \mathbb{R}^N \to \mathbb{R}_+$ as (29) with

$$\lambda_i(s) = \left[\frac{1}{\tau}\alpha_i(s)\right]^{\varphi} \left[\sigma_{3-i}(s)\right]^{\varphi+1}, \ i = 1, 2 \tag{107}$$

which are of class $\mathcal{K}$ and satisfy $\lambda_i'(s) \ge 0$ for all $s \in \mathbb{R}_+$. For these functions we obtain

$$\lambda_i'(s) = \frac{1}{\tau}\left[\frac{1}{\tau}\alpha_i(s)\right]^{\varphi-1} \left[\sigma_{3-i}(s)\right]^{\varphi} \cdot \left[\varphi\alpha_i'(s)\sigma_{3-i}(s) + (\varphi+1)\alpha_i(s)\sigma_{3-i}'(s)\right], \text{for } \varphi > 0 \tag{108}$$

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