Errata for page 1517

\[
P\{\|x(t)\| < \infty, \forall t \in \mathbb{R}_+\} = 1, \quad \forall x(0) \in \mathbb{R}^N, \quad (58)
\]
\[
P\left\{\frac{\alpha(|x(t)|)}{1 + \int_0^t \nu(|r(\tau)|) d\tau}\right\} \geq 1 - \epsilon, \quad \forall t \in [0, \xi], \quad \forall x(0) \in \mathbb{R}^N \setminus \{0\}, \quad \forall \epsilon \in (0, 1). \tag{59}
\]

\[
E[V(x(t_A \wedge t))] \leq V(x(0)) + e(t) \tag{62}
\]
follows from $\mathcal{L}V \leq \sigma(|r|)$. Using $P\{t_A \leq t\} \inf_{|y| \geq A} V(y) \leq E[V(x(t_A \wedge t))]$ implied by (61), from (62) we obtain
\[
P\{t_A \leq t\} \leq \frac{V(x(0)) + e(t)}{\alpha(A)}. \tag{63}
\]

(5). By definition, we have $v(0) = V(x(0)) \leq z(0)$ and $v(t) \geq 0$ for all $t \in \mathbb{R}_+$. Given $l \in \mathbb{R}_+$, for each $\epsilon$, $x(0)$ and $r$, define $T(l) \in [0, \infty]$ as
\[
T(l) := \inf \{t \geq 0 : v(t) \geq z(l)\}, \tag{65}
\]
given $\epsilon$, $x(0)$ and $r$ it holds for each $l \in \mathbb{R}_+$ that
\[
\{T(l) \leq t\} \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_+. \tag{66}
\]
Thus, applying the argument of [17, Proof of Lemma 3.2, p.73] to the stopped process $x(T \wedge t)$ with (66), we obtain
\[
E[v(T \wedge t)] = V(x(0)) + E\left[\int_0^{T \wedge t} \mathcal{L}V(x(\tau)) d\tau\right]
\]
for each $t \in \mathbb{R}_-$. Property $\mathcal{L}V \leq \sigma(|r|)$ yields
\[
E[v(T \wedge t)] \leq V(x(0)) + e(t) \tag{67}
\]
since $T \wedge t \leq t$. The definition of $T$ and $v(t) \geq 0$ yield
\[
E[v(T \wedge t)] \geq E[I_{\{T \leq t\}} v(T)] = z(t) P(T \leq t), \tag{68}
\]
where $I_{\{T \leq t\}}$ is the indicator function of the set $\{T \in \mathbb{R}_+ : T \leq t\}$. Combining (68) with (67) yields
\[
V(x(0)) + e(t) \geq z(t) P\{T(l) \leq t\}. \tag{69}
\]
for each $l \in \mathbb{R}_+$. Substituting (64) into (69) gives
\[
\epsilon \geq P\{T(l) \leq t\}, \quad \forall t \in [0, \xi]. \tag{70}
\]
By virtue of $T$ defined in (65) with (60) and (64) and the property $\alpha(|x(0)|) \leq V(x(t)) = v(t)$, using (70), we arrive at (59).

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The components $w_e$ of $w \in \mathbb{R}^n$ are again mutually independent standard Wiener processes. The $(k, k)$-component of $\theta$ represents the intensity describing the influence of the $k^{th}$ component of $u(t)$ on $z(t)$ through the $k^{th}$ column of $h(z)$. In fact, the deterministic function $\theta(t)(h_t(\cdot))$ is the infinitesimal variance matrix of the $S$-dimensional stochastic process represented by $\theta(t)dz(t)$ in (10). We assume $\theta(t) = 0$. It is stressed that for (10), we do not assume $h(0) = 0$. This paper employs the notion of noise-to-state stability for system (10) introduced in [18].

**Definition 5:** System (10) is said to be noise-to-state stable (NSS) if for each $e > 0$, there exist a class $\mathcal{KL}$ function $\gamma_1$ and a class $\mathcal{K}$ function $\gamma_2$ such that

$$
\gamma_1(\rho_t) < \gamma_2(\Theta_t(0), t) + \left( \sup_{\rho \in \partial \Omega} \gamma_1(\theta(t) \Theta_t^{-1}(\rho)^D_t) \right) e^t
$$

for all $t$, $\rho \in \mathbb{R}_+$. $\Theta_t(0) = 0$. (11)

NSS defines robustness with respect to the noise variance $\Theta(t)^D_t$. This idea contrasts with the one employed by another type of ISS proposed and investigated in [28], [31], where $r(t)$ is a random variable in addition to the Wiener process $w(t)$. As we did for (5), we define the following two properties for system (10):

**Definition 6:** System (10) is said to be integral noise-to-state stable (INSS) if for each $e > 0$, there exists a class $\mathcal{KL}$ function $\bar{\gamma}_1$, a class $\mathcal{K}$ function $\bar{\gamma}_2$, and a class $\mathcal{K}$ function $\bar{\chi}$ such that

$$
\bar{\gamma}_1(\rho_t) < \gamma_2(\Theta_t(0), t) + \left( \sup_{\rho \in \partial \Omega} \bar{\gamma}_1(\theta(t) \Theta_t^{-1}(\rho)^D_t) \right) e^t
$$

for all $t$, $\rho \in \mathbb{R}_+$. $\Theta_t(0) = 0$. (12)

**Definition 7:** System (10) is said to be quasi-integrable noise-to-state stable (QINSS) if there exists a constant $\Gamma > 0$ satisfying the following: for each $e > 0$, there exists a class $\mathcal{KL}$ function $\gamma_1$, a class $\mathcal{K}$ function $\gamma_2$, and a class $\mathcal{K}$ function $\gamma_3$ such that

$$
\gamma_1(\rho_t) < \gamma_2(\Theta_t(0), t) + \left( \sup_{\rho \in \partial \Omega} \gamma_3(\theta(t) \Theta_t^{-1}(\rho)^D_t) \right) e^t
$$

for all $t$, $\rho \in \mathbb{R}_+$. $\Theta_t(0) = 0$. (13)

In Definitions 5-7, we do not require the influence of the

**IV. LYAPUNOV CHARACTERISTICS**

For any given $C^2$ function $V : \mathbb{R}^n \rightarrow [V(x)]$, the infinitesimal generator $A_z$ associated with systems (3) and (10) is defined as

$$
\mathcal{L}V = \frac{\partial V}{\partial x} A_z + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} Q(z) \frac{\partial^2 V}{\partial x_i \partial x_j} M_p(z)^{-1} M_p(z)^{-1}
$$

where

$$
\begin{align*}
Q &= f(z) \\
Q &= \sigma(z) \quad (\text{for (3) and (5)})
\end{align*}
$$

(15)

Here, the symbol $I$ denotes the identity matrix of size $S \times S$.

**A. Robustness With Respect to Deterministic Disturbance**

For ISS, the following characterization is available, which is parallel to the deterministic case [27].

**Proposition 1:** Consider (5). If there exist a positive definite and radially unbounded $C^2$ function $V : \mathbb{R}^n \rightarrow [V(x)]$, and continuous functions $\mu_1$ and $\mu_2$ in $\Theta$ such that the implication $V(x) \geq \mu_1(\rho_t) \Rightarrow \mathcal{L}V \leq \mu_2(\rho_t)$ holds for all $x \in \mathbb{R}_+^n$, then system (5) is ISS in probability.

As indicated in [29], the proof of Proposition 1 essentially follows an adaptation of the one given in [18] which is demonstrated in detail in [20]. Note that applying [17, Theorem 5.1] or, [21, Theorem 2.4 in Section 4.2] to the proof of [18, Theorem 3.3] allows us to replace $\Theta$ with $\rho \in \mathbb{R}^n$. A related discussion on Proposition 1 is given in Appendix H. The main developments in this subsection are the following two theorems establishing quasi-ISS and ISS in probability.

**Theorem 1:** Consider (5). If there exist a positive definite and radially unbounded $C^2$ function $V : \mathbb{R}^n \rightarrow [V(x)]$, and continuous functions $\mu_1$ and $\mu_2$ in $\Theta$ such that

$$
\mathcal{L}V \leq -\mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)

holds for all $x \in \mathbb{R}_+^n$ and $\rho \in \mathbb{R}^n$, then system (5) is ISS in probability.

In [29], [33], and [34], the function $\rho_0$ in (18) is assumed to be of class $\mathcal{K}$, in order to obtain ISS of (5). Indeed, if $\rho_0 \in \mathcal{K}$ holds, inequality (17) is satisfied for any $\tau > 1$ with $\rho_0 \subset \tau \subset \mathcal{K}$ and $\eta \left( 1 / \rho_0 \right) \subset \mathcal{K}$. As in the deterministic case, we can relax $\rho_0$ into $\mathcal{K}$.

**Theorem 2:** Consider (3). If there exist $D > 0$ and $\Omega$ such that

$$
\mathcal{L}V \leq -\mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)

holds, then system (3) is ISS in probability.

For notational simplicity, the above theorem employed the following notation:

$$
\begin{align*}
\lim_{\rho \to 0} & \rho_0 \rho \to 0, \quad \rho_0 \subset \mathcal{K} \quad (18)
\end{align*}
$$

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Therefore, the function $V$ satisfying (18) establishes not only quasi-ISS but also ISS of the stochastic system (5) in both cases of $\rho_0 \in \mathcal{K}$ and (19). In the deterministic case, the function $\rho = \rho_0 + \tau \rho_0$ for a constant $\tau > 1$ resp. a continuous function $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying

$$
\mathcal{L}V \leq \mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)

holds for all $x \in \mathbb{R}_+^n$, then system (5) is ISS in probability.

A function $V$ satisfying the conditions of Precisive 2 is

$$
\mathcal{L}V \leq \mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)

for all $x \in \mathbb{R}_+^n$, then system (5) is ISS in probability.

A function $V$ satisfying the conditions of Precisive 2 is

$$
\mathcal{L}V \leq \mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)

for all $x \in \mathbb{R}_+^n$, then system (5) is ISS in probability.

A function $V$ satisfying the conditions of Precisive 2 is

$$
\mathcal{L}V \leq \mu_1(\rho_t) + \mu_2(\rho_t)
$$

(17)
of the k-th component of $u(t)$ on $x(t)$ through the k-th column of $h_k(x)$. The matrix $H_k(x)$ is obtained as

$$H_k(x) = \left[ \begin{array}{c} e_1(x) \\ \vdots \\ e_k(x) \end{array} \right].$$

It is stressed that in contrast to (26) and (27), we do not assume $h_k(0) = 0$ for (44) and (45). The following is assumed throughout this subsection.

Assumption 2: For each $i = 1, 2$, there exist a positive definite and radially unbounded $C^2$ function $v_i: \mathbb{R}^n \to \mathbb{R}_+$, a $C^2$ function $\alpha_i, \beta_i, \kappa_i \in \mathbb{K}$ and a $C^0$ function $\varphi_i \in \mathcal{K}(0)$ such that

$$L_i^2 \leq -\alpha_i\varphi_i^2(x_i(t)) + \sigma_i(x_i(t)) + \omega_i(x_i(t)),$$

(46)

holds for all $x_i \in \mathbb{R}^n$, $r_i \in \mathbb{R}^n$, and all $\theta_i \in \mathbb{R}^n$, where $\omega_i$ is the $i$-th function, i.e., $\omega_i = 0$ if $i = 0$. Here, is satisfied, then the following hold true:

(1) If

$$\lim_{x \to \infty} \frac{\alpha_i}{\beta_i} \frac{\varphi_i}{\sigma_i} (x_i(t)) = 0,$$

(51)

then $h_k(x)$ is determined, interconnection (44), (45) is quasi-NSIS.

(2) If there exists $D > 0$ such that (36) is satisfied, interconnection (44), (45) is NSIS.

(3) If there exist $D > 0$, $i = 1, 2$, such that (37) and (38) are satisfied, interconnection (44), (45) is NSIS.

(4) If $\alpha_i$ and $\beta_i$ are of class $\mathcal{K}_\infty$ and (51) hold, interconnection (44), (45) is NSIS.

The difference from quasi-NSIS, NSIS and NSS appears in

$$\phi((h)(m)) \varphi_i(x_i(t)), \ \forall x_i \in [0, \infty),$$

From (18) and property (23) in (73) it also follows that

$$W_i(t) \geq D \frac{d}{dt} W_i(t) \leq -\alpha_i(x_i(t)) + \beta_i(x_i(t)).$$

(75)

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From the above and property (23) in (75) it also follows that

$$W_i(t) \geq \alpha_i(x_i(t)) + \beta_i(x_i(t)).$$

(76)

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Thus, Theorems 1 completes the proof.

$$\|\varphi_i(t)\| \leq \beta_i(1 + \frac{\gamma}{\alpha_i}) \|x_i(t)\|$$

(103)

by dividing the evaluation of the above $L_i^2$ through the two cases, $\alpha_i(\varphi_i(t)) \leq \gamma \|\varphi_i(t)\|$ and $\alpha_i(\varphi_i(t)) > \gamma \|\varphi_i(t)\|$, respectively. Let $\sigma_i \in \mathbb{K}_\infty$ and $\Gamma$ be

$$\hat{\sigma}(x) = \max \{\hat{\sigma}_1(x), \hat{\sigma}_2(x)\}, \ \forall x \in \mathbb{R}^n,$$

and define $\hat{D}: \mathbb{R}^n \to \mathbb{R}^n$ by $\hat{D}(x) = \hat{\sigma}(x) + \hat{\Gamma}(x)$ which is continuous and non-decreasing in both $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. By virtue of the continuity of trajectories, combining (101), (102), and (103) yields

$$P \|V(x(t))\| \leq \hat{D}(\|V(x(t))\|), \ \forall x \in \mathbb{R}^n,$$

(104)

P \|V(x(t))\| \leq \hat{D}(\|V(x(t))\|), \ \forall x \in \mathbb{R}^n,$$

(104)

Thus, Theorems 1 completes the proof.

$$\|\varphi_i(t)\| \leq \beta_i(1 + \frac{\gamma}{\alpha_i}) \|x_i(t)\|$$

(103)

by dividing the evaluation of the above $L_i^2$ through the two cases, $\alpha_i(\varphi_i(t)) \leq \gamma \|\varphi_i(t)\|$ and $\alpha_i(\varphi_i(t)) > \gamma \|\varphi_i(t)\|$, respectively. Let $\sigma_i \in \mathbb{K}_\infty$ and $\Gamma$ be

$$\hat{\sigma}(x) = \max \{\hat{\sigma}_1(x), \hat{\sigma}_2(x)\}, \ \forall x \in \mathbb{R}^n,$$

and define $\hat{D}: \mathbb{R}^n \to \mathbb{R}^n$ by $\hat{D}(x) = \hat{\sigma}(x) + \hat{\Gamma}(x)$ which is continuous and non-decreasing in both $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. By virtue of the continuity of trajectories, combining (101), (102), and (103) yields

$$P \|V(x(t))\| \leq \hat{D}(\|V(x(t))\|), \ \forall x \in \mathbb{R}^n,$$

(104)