State-Dependent Scaling Problems and Stability of Interconnected iISS and ISS Systems

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***Abstract*— This paper addresses the problem of establishing stability of nonlinear interconnected systems. This paper introduces a mathematical formulation of the state-dependent scaling problems whose solutions directly provide Lyapunov functions proving stability properties of interconnected dissipative systems in a unified manner. Stability criteria are interpreted as sufficient conditions for the existence of solutions to the state-dependent scaling problems. Computing solutions to the problems is straightforward for systems covered by classical stability criteria. It, however, could be too difficult for systems with strong nonlinearity. The main purpose of this paper is to demonstrate the effectiveness beyond formal applicability by focusing on interconnected integral input-to-state stable(iISS) systems and input-to-state stable(ISS) systems. This paper derives small-gain-type theorems for interconnected systems involving iISS systems from the state-dependent scaling formulation. This paper provides solutions and Lyapunov functions explicitly. The new framework seamlessly generalizes the ISS small-gain theorem and classical stability criteria such as the $\mathcal{L}_p$ small-gain theorem, the passivity theorems, the circle and Popov criteria. State-dependence of the scaling is crucial for effective treatment of essential nonlinearities, while constants are sufficient for classical nonlinearities.**

***Index Terms*— Nonlinear interconnected system, dissipation, Lyapunov function, integral input-to-state stability, input-to-state stability. small-gain condition**

I. INTRODUCTION

IN the literature of nonlinear control theory, a great deal of effort has been put into the problem of finding useful formulations of conditions under which interconnected systems are stable. One of significant contributions is the stability theory developed in [1], which unifies previously known stability criteria and provides Lyapunov versions of input-output stability results such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, and the circle and Popov criteria. Another major development which presently plays an important role in nonlinear control analysis and design is the ISS small-gain theorem also known as the nonlinear small-gain theorem[2], [3]. A small-gain theorem which brought about the ISS small-gain theorem was originally formulated by Hill[4], and Mareels and Hill[5], and that was extended in the ISS framework by Jiang et al.[2] which was also further generalized by Teel[3]. The effectiveness of the ISS small-gain theorem is evident when systems have essential nonlinearities described by the input-to-state stable(ISS) property[6]. It is, however, known that there are systems for which ISS is too strong requirement[7], [8]. One has yet to develop a stability theory which encompasses much broader classes of interconnected systems. For nonlinear systems, universal applicability and effectiveness do not come together automatically. This is a reason why there are two directions of the research. One direction pursues problem-specific techniques focusing on particularity of individual nonlinearities. Some people consider them too heuristic and impractical even when specialized tricks are effective. The other direction seeks general techniques that are applicable to many cases in a unified way. The generality sometimes not only excludes some strong nonlinearities of great importance, but also renders the essential effectiveness obscure so that the applicability is only formal. It is typical of general 'nonlinear' problems to have no guarantee of the existence of solutions. We often do not know how to solve them even if solutions exist. Naturally, this situation has brought out a quest for a successful fusion of the two directions. From this viewpoint, it is remarkable that the ISS small-gain theorem achieves a balance between the universal applicability and the effectiveness for interconnected ISS systems[9], [10], [11].

The aim of this paper is to provide a general framework which is not limited to the settings of popular classical stability criteria and the ISS small-gain theorem. To this end, this paper borrows an idea from the state-dependent scaling techniques which have been recently introduced by the author[12], [13], [14] for constructing robust control Lyapunov functions for some classes of systems. This paper generalizes the idea much further. Problems of stability analysis for interconnected dissipative systems are formulated into state-dependent scaling problems in a unified way. This paper clarifies for the first time the relation between the state-dependent scaling formulation and the ISS small-gain condition[2], [3] as well as stability criteria for dissipative systems[1]. The state-dependent scaling problems are scalar inequalities we solve for parameters which the author calls state-dependent scaling functions. Solutions immediately lead to Lyapunov functions for interconnected systems.

The state-dependent scaling approach this paper pursues is a tool of dealing with nonlinear systems beyond classical nonlinearities such as finite linear-gain, sector and passivity-related systems which have been popular in textbooks of nonlinear stability analysis. This paper is devoted mainly to demonstration of the effectiveness much more than formal applicability by concentrating on the interconnected system composed of integral input-to-state stable(iISS) and ISS systems. The existence of solutions to the state-dependent scaling problem is investigated rigorously, and explicit formulas of the solutions are shown. New theorems of the small-gain-type are derived. To the best of author's knowledge, the result of

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small-gain-type theorems involving iISS systems is the first of its kind. The class of ISS systems has been extensively investigated and has been playing an important role[6], [15], [9]. For instance, the fact that cascades of ISS systems are ISS is widely used in stabilization. The ISS small-gain theorem is also a popular tool for feedback interconnection. In contrast, the concept of iISS has not yet been fully exploited in analysis and design although the property of iISS by itself has been investigated deeply[7], [8]. The iISS property covers nonlinearities much broader than the ISS property. Indeed, the iISS captures important characteristics essentially nonlinear systems often have[8], and there are many practical systems which are iISS, but not ISS. There are still few tools of making full use of the iISS property in systems analysis and design. For instance, stability criteria similar to the ISS small-gain theorem have not been developed for interconnection involving iISS systems so far.

The paper places a special emphasis on construction of Lyapunov functions for interconnected systems. Storage functions in the dissipative analysis[16], [1] often serve as Lyapunov functions. The storage function is an abstract notion of energy stored in a system. The energy increases when energy is supplied from outside. The supply rate determines the variation of the storage function. The idea proposed in [1] is to construct a Lyapunov function of interconnected systems explicitly by summing up supply rates of individual systems. On the other hand, the ISS small-gain theorem are usually explained in terms of trajectories of systems (in other words, input-output-type formulation)[2], [3], [9]. Although the ISS property of open-loop systems has been related to Lyapunov functions[17], [18], [9], little development has been made in the construction of Lyapunov functions for feedback systems. A notable exception is [19] which proves the equivalence between gain-type formulation and Lyapunov-type formulation of the ISS small-gain theorem. It, however, focuses on the equivalence rather than in providing explicit formulas for Lyapunov functions which are convenient for further use. In order to close the remaining gap between the dissipative approach and the ISS small-gain theorem, this paper comes up with an idea of summing up supply rates nonlinearly for constructing Lyapunov functions for interconnected systems. The nonlinearly-scaled sum of supply rates and several existing tools share a common tool of integration to rescale storage or Lyapunov functions. The above-mentioned papers[18], [9] exploit integrals of Lyapunov functions for cascades of ISS systems. Mazenc and Praly[20] address a class of nonlinear systems in feedforward form using integral for rescaling Lyapunov functions. This paper investigates a generalized usage of such a technique and pursues its real potential further for new classes of systems. Nonlinear coefficients to combine supply rates for the ISS small-gain theorem and constant coefficients for the dissipative approach are solutions to the state-dependent problems. Such formulation enables us to not only explain the dissipative approach and the ISS small-gain theorem in a unified language, but also treat iISS systems successfully in the same framework.

This paper is organized as follows. First, Section II gives a glimpse into this paper. Section III formulates the problem of stability of interconnected systems in a general configuration. Then, the section introduces a mathematical problem of state-dependent scaling which is the preliminary idea of this paper. A Lyapunov function establishing stability of the interconnection is obtained explicitly from a solution to the state-dependent scaling problem. For interconnected systems covered by traditional stability criteria, such as the $\mathcal{L}_p$ small-gain theorem, the passivity theorems, and the circle and Popov criteria, solutions can be computed easily as positive constants, and the stability criteria are simply sufficient conditions for the existence of the solutions. For arbitrarily general systems, formulas and existence conditions have not been available so far. Therefore, this paper is mainly devoted to the issues of when the solutions exist and how they can be found for systems which are not covered by the classical stability criteria. Section III focuses on the interconnection of iISS systems and ISS systems, and provides explicit formulas. Small-gain-like conditions are derived as sufficient conditions for guaranteeing the existence of solutions to the state-dependent scaling problem. Section IV proposes an extension of the state-dependent scaling problem to obtain less conservative stability formulas in the presence of static systems. In Section V, the effectiveness of the proposed approach is illustrated through examples. Finally, conclusions are drawn in Section VI.

This paper uses the following notations. The interval $[0,\infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. Euclidean norm of a vector in $\mathbb{R}^n$ of dimension n is denoted by $|\cdot|$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be class $\mathcal{K}$ and written as $\gamma\in\mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0)=0$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be class $\mathcal{K}_\infty$ and written as $\gamma\in\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty}\gamma(r)=\infty$. We write $\gamma\in\mathcal{P}_0$ for a function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ if it is a continuous function satisfying $\gamma(0)=0$. The set of $\gamma\in\mathcal{P}_0$ satisfying $\gamma(s)>0$ for all $s\in\mathbb{R}_+\setminus\{0\}$ is denoted by $\gamma\in\mathcal{P}$.

II. Big Picture

The purpose of this introductory discussion is to give the reader motivation for going through details of this paper. For establishing stability of interconnected systems, this paper proposes state-dependent scaling formulation applicable to general systems which possess dissipation property. Naturally, a question arises as to whether the state-dependent scaling problems to be formulated are solvable practically. The answer is affirmative for finite linear-gain nonlinearities, sector and passive systems popular in nonlinear stability textbooks. Indeed, solutions of the state-dependent scaling problems are constants for such classes of systems, and the formulation reduces to linear combination of supply rates. The constant scaling (or equivalently, linear combination of supply rates) classical stability criteria rely on fail easily in solving stability problems for strongly nonlinear systems. In order to go beyond existing approaches, the paper will rigorously investigate new classes of nonlinear interconnection involving iISS subsystems, and substantiate the effectiveness of the state-dependent scaling approach by providing affirmative answers to the question.

Nevertheless, there are still practical systems which do not fit in any supply rates of the iISS property and others covered by existing approaches. The classes of ISS and iISS systems are described by supply rates which generalize the notion of gain by incorporating nonlinear functions into the input and the output. As we will show, the state-dependent scaling handles the nonlinear variation and the mismatch of nonlinearities between subsystems. The state-dependent scaling can also allow us to select fictitious input and output incorporating nonlinear variations into supply rates of the passive type. There are interconnected systems whose stability can be established by exploiting passivity, gain and their nonlinear variations at the same time, although the stability cannot be established by using any one of them alone. The state-dependent scaling approach to be proposed in this paper is suitable for those systems since it puts passivity, gain and their nonlinear variations into a unified formulation.

Suppose that an interconnected system consists of two subsystems satisfying dissipation inequalities

$$\Sigma_1 : \ \dot{V}_1(x_1) \le -x_1^2 + x_1^3 x_2^3 \quad (1)$$
$$\Sigma_2 : \ \dot{V}_2(x_2) \le -x_2^4 - x_2 x_1^3 + x_2 x_1 \quad (2)$$

for storage functions $V_i(x_i) = x_i^2/2$, $i = 1, 2$. The second subsystem Σ_2 is ISS since we obtain

$$\dot{V}_2(x_2) \le -\frac{1}{2}x_2^4 + \frac{3}{4}x_1^4 + \frac{3}{4}x_1^{4/3} \quad (3)$$

by using Young's inequality. On the other hand, the subsystem Σ_1 is neither ISS nor iISS. Indeed, the differential equation

$$\dot{x}_1 = -x_1 + x_1^2 x_2^3 \quad (4)$$

satisfying (1) is not ISS with respect input x_2 and state x_1. Furthermore, even for exponentially decaying input $x_2(t)$, the equation (4) has solutions escaping to infinity in finite time[27]. In the presence of the subsystem Σ_2 having such a serious finite escape time property, the dissipation inequality (3) and stability criteria based on the gain are not helpful. Passivity theorems cannot establish global asymptotic stability of the interconnected system either since there are no terms in (1) which cancel $+x_2x_1$ in (2). As we will show, if appropriate functions λ_1 and λ_2 can be found, where λ_1 and λ_2 are continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, that satisfy

$$\begin{aligned}&\lambda_1(V_1(x_1))\{-x_1^2 + x_1^3 x_2^3\}\\ &\quad +\lambda_2(V_2(x_2))\{-x_2^4 - x_1^3 x_2 + x_1 x_2\}\\ &\qquad \le \rho_e(x_1, x_2), \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \end{aligned} \quad (5)$$

for some function ρ_e which is continuous and strictly negative, then the interconnection of (1) and (2) has a globally asymptotically stable equilibrium at the origin. The equation (5) will be called a state-dependent scaling problem and λ_1 and λ_2 are scaling functions which combine supply rates of the two subsystems. In order to have the inequality (5) satisfied, it is not difficult too see that the positive component generated by the cubic term x_1^3 in (1) must be dominated by the negative component generated by $-x_1^3$ in (2). Since the other terms do not have suitable growth order with respect to x_1 and the function $x_1^3 x_2^3$ is not sign definite, the inequality (5) is satisfied only if

$$\lambda_1(V_1(x_1))\{x_1^3 x_2^3\} + \lambda_2(V_2(x_2))\{-x_1^3 x_2\} = 0 \quad (6)$$

holds. The solution to this equation is

$$\lambda_1(V_1(x_1)) = 1, \quad \lambda_2(V_2(x_2)) = x_2^2 \quad (7)$$

Substituting this pair into the left hand side of (5), we obtain

$$-x_1^2 + x_1^3 x_2^3 - x_2^6 - x_1^3 x_2^3 + x_1 x_2^3 \le -x_1^2 - x_2^6 + x_1 x_2^3$$

Thus, the inequality (5) is achieved with

$$\rho_e(x_1, x_2) = -(x_1^2 + x_2^6)/2$$

Hence, global asymptotic stability of the equilibrium at the origin is established for the interconnection of (1) and (2).

Although the above is only a very simple example, the unified treatment of various properties composing dissipation inequalities and the nonlinearity of scaling are the key to the success.

III. State-Dependent Scaling Problem and Solutions

In this paper, two mathematical problems play a central role in establishing stability properties and constructing Lyapunov functions of nonlinear interconnected systems. This section presents one of the two problems and explains the main idea of the framework this paper proposes.

A. Problem Formulation

Consider the interconnected system Σ shown in Fig.1. This paper deals with subsystems Σ_1 and Σ_2 of the general form

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \quad (8)$$
$$\Sigma_2 : \ \dot{x}_2 = f_2(t, x_2, u_2, r_2)\ . \quad (9)$$

These dynamic systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. The state vector of the interconnected system Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of Σ_i. It is assumed that $f_1(t, 0, 0, 0) = 0$ and $f_2(t, 0, 0, 0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \ge 0$. The functions f_1 and f_2 are assumed to be piecewise continuous in t, and locally Lipschitz in the other arguments. These restrictions on f_i are only for assuming the existence of a unique maximal solution of the initial value problem $\dot{x}_i = f_i$ for each $u_i \in \mathbb{R}^{p_i}$ and $r_i \in \mathbb{R}^{m_i}$. This paper does not assume that f_i describing Σ_i in (8) and (9) are precisely known. They are left unknown. Instead, we associate Σ_1 and Σ_2 in Fig.1 with supply rates and assume the knowledge of dissipation inequalities as follows.

Assumption 1: There exists a $\mathbf{C}^1$ function $V_i : (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t, x_i) \le \bar{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, \ t \in \mathbb{R}_+ \quad (10)$$

holds with $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$, and

$$\frac{dV_i}{dt} \le \rho_i(x_i, u_i, r_i), \quad \forall x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{p_i}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+ \quad (11)$$

holds along the trajectories of the system Σ with a continuous function $\rho_i : (x_i, u_i, r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \times \mathbb{R}^{m_i} \to \mathbb{R}$ satisfying $\rho_i(0, 0, 0) = 0$.

A system Σ_i satisfying Assumption 1 is said to be dissipative with respect to storage function V_i and supply rate ρ_i[16], [1], [21].

This section formulates stability of the interconnected system into a mathematical problem which this paper refers to as Problem 1.

Problem 1: Given continuously differentiable functions $V_i : (t,x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ and continuous functions $\rho_i : (x_i,x_j,r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{m_i} \to \mathbb{R}$ for $i=1,2$ and $j=\{1,2\}\setminus\{i\}$, find continuous functions $\lambda_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lambda_i(s) > 0 \quad \forall s \in (0,\infty) \tag{12}$$

$$\lim_{s\to 0^+} \lambda_i(s) < \infty \tag{13}$$

$$\int_1^\infty \lambda_i(s)ds = \infty \tag{14}$$

for $i=1,2$ such that

$$\begin{aligned}\lambda_1(V_1(t,x_1))\rho_1(x_1,x_2,r_1) + \lambda_2(V_2(t,x_2))\rho_2(x_2,x_1,r_2) \\ \leq \rho_e(x_1,x_2,r_1,r_2), \\ \forall x_1\in\mathbb{R}^{n_1}, x_2\in\mathbb{R}^{n_2}, r_1\in\mathbb{R}^{m_1}, r_2\in\mathbb{R}^{m_2}, t\in\mathbb{R}_+\end{aligned} \tag{15}$$

holds for some continuous function $\rho_e : (x_1,x_2,r_1,r_2) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_1,x_2,0,0) < 0 \quad ,\forall (x_1,x_2) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\setminus\{(0,0)\} \tag{16}$$

The following theorem motivates the above mathematical problem.

Theorem 1: The equilibrium $x=0$ of the interconnected system Σ given by (8) and (9) is globally uniformly asymptotically stable for $r(t)\equiv 0$ if there is a solution $\{\lambda_1,\lambda_2\}$ to Problem 1. Furthermore, a $\mathbf{C}^1$ function $V_{cl} : (t,x) \in \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}_+$ defined with the solution $\{\lambda_1,\lambda_2\}$ as

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{17}$$

satisfies

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|), \quad \forall x\in\mathbb{R}^n, t\in\mathbb{R}_+ \tag{18}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ and

$$\frac{dV_{cl}}{dt} \leq \rho_e(x,r), \quad \forall x\in\mathbb{R}^n, r\in\mathbb{R}^m, t\in\mathbb{R}_+ \tag{19}$$

holds along the trajectories of the system Σ.

The function defined in (17) serves as a Lyapunov function of the interconnected system Σ. It may be worth mentioning that (13) is redundant mathematically since each λ_i is supposed to be continuous on $\mathbb{R}_+ = [0,\infty)$. The explicit statement may direct the readers' attention to it.

The solutions λ_i to the inequality of *the sum of scaled supply rates* (15) immediately lead us to Lyapunov functions establishing the stability of the interconnected system Σ. The parameters λ_1 and λ_2 scaling supply rates in (15) are functions of the state variables x_1 and x_2. More precisely, they are functions of V_i depending on x_i as in (10). An extended formulation of Problem 1 whose detailed introduction is postponed until Section IV allows λ_1 to directly depend on x_1. The pair of Problem 1 and the extension(to be called Problem 2) can be regarded as a general formulation of the state-dependent scaling technique [12], [13], [14]. Thus, this paper referred to Problem 1 and Problem 2 as state-dependent scaling problems. The functions λ_i are referred to as state-dependent scaling functions. Early results developed in [12], [13], [14] were based on some special cases of Problem 1 and Problem 2 where the supply rates are restricted to finite $\mathcal{L}_2$-gain, ISS or a subset of iISS. Those papers originally referred to $1/\lambda_i$ as the state-dependent scaling factors.

Solutions to the state-dependent scaling problems also establish stability properties of cascade systems under Assumption 1. Indeed, if one of feedback paths $u_1 = x_2$ and $u_2 = x_1$ is disconnected in Fig.1, the interconnection becomes a cascade. When the path of u_i is disconnected, the supply rate $\rho_i(x_i,u_i,r_i)$ becomes $\rho_i(x_i,r_i)$. By the cascade system Σ_c, the paper means that the path of $u_1 = x_2$ is cut, which is depicted in Fig.2.

Remark 1: When Σ_1 and Σ_2 are time-invariant, the consequence of Theorem 1 is true even if '$<$' in (16) is replaced by '$\leq$' on appropriate assumptions of zero-state detectability[1].

Remark 2: For supply rates ρ_i popular in classical stability analysis, the state-dependent scaling formulation reduces to well-known techniques. Classical stability criteria can be considered as sufficient conditions for the existence of solutions to Problem 1 for finite linear-gain nonlinear systems, sector and passivity-related systems[1], and solutions can be obtained directly from those classical techniques[22]. Indeed, the formulation of the state-dependent scaling problems smoothly extends stability criteria unified in [1] (presented in standard textbooks such as [23], [24]) and passivity theorems discussed in [25]. It is not difficult to see that all stability theorems based on the early works on interconnected dissipative systems[16], [1], [21] are explained by 'constant' parameters λ_1 and λ_2, namely, the state-dependent scaling problems are in the form of

$$\lambda_1\rho_1(x_1,x_2) + \lambda_2\rho_2(x_2,x_1) \leq \rho_e(x) \ .$$

Thus, the $\mathcal{L}_p$ small-gain theorem, the passivity theorems, the Popov and circle criteria, are proved by using linear combinations of supply rates. By contrast, the inequality (15) of Problem 1 is not a linear combination of supply rates. The parameters λ_1 and λ_2 are allowed to be functions. The exploitation of state-dependence or nonlinearities in those parameters is naturally vital for dealing with strong nonlinearities which are not covered by the classical stability criteria.

Remark 3: The use of integral in Lyapunov functions can be found in existing papers. This paper investigates a generalized usage and exploits the potential of the technique further. In [18], [9], given a system, the integral is introduced for changing a supply rate into a desirable one and applied elegantly to stability of cascade systems. The formulation proposed in this paper aims at exploiting integrands flexibly to obtain stability of general interconnected systems directly from given supply rates without transformation into particular forms. Mazenc and Praly[20] also deals with a class of

[1]The relation between Problem 1 and the stability involving static subsystems is discussed in Section IV.

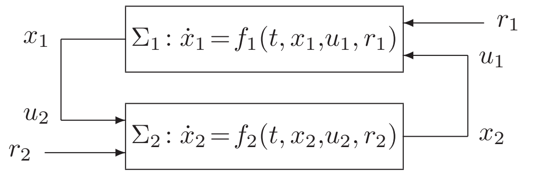

Fig. 1. Feedback interconnected system Σ

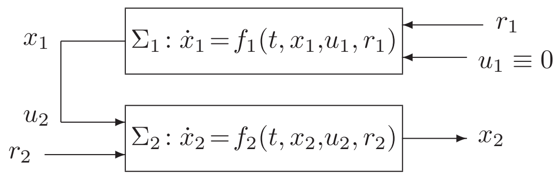

Fig. 2. Cascade system Σ_c

feedforward systems using Lyapunov functions with integral scalings. This paper suggests that the use of those integral Lyapunov functions is regarded as a restricted subset of a broader problem of Problem 1, which is generalized down to the state-dependent scaling formulation.

The state-dependent scaling problem is directly related to construction of Lyapunov functions. The formulation only requires systems to be dissipative, so that it covers a broader class of systems than classically popular stability criteria[1], [23], [24]. This subsection has not mentioned how easy or difficult it is to find solutions to state-dependent scaling problems for systems which are not covered by traditional supply rates[1], [23], [24], [25]. The state-dependent scaling problem is jointly affine in the scaling functions λ_1 and λ_2. This affine property should be helpful in calculating solutions, which is the main issue investigated in the remaining part of this section.

B. Supply Rates of iISS and ISS Systems

The rest of this section aims to show explicit solutions to the state-dependent scaling problem for supply rates characterizing stronger nonlinearities than traditional ones. For this purpose, this paper focuses on iISS and ISS types of supply rates. Concepts of iISS and ISS properties were introduced by Sontag[6], [7], and the class of iISS systems is broader and includes stronger nonlinearities than the class of ISS systems. In subsequent subsections, we derive small-gain rules as conditions guaranteeing the existence of solutions to Problem 1 for iISS and ISS properties. It is the first formulation of its type to address stability of interconnection involving iISS systems. The ISS small-gain condition[2], [3] is explained as a special case dealing with interconnection of ISS systems.

In order to concentrate on ISS and iISS properties[6], [7], we assume that a supply rate function of the form

$$\rho_i(x_i, u_i, r_i) = -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{20}$$

is given for each Σ_i, $i = 1, 2$. In the case that the second input r_i is null, the function σ_{ri} vanishes. As we did in the previous subsection, it is assumed that α_i, σ_i ,$\sigma_{ri} \in \mathcal{P}_0$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ satisfying Assumption 1 are known. Let us consider the following four sets of supply rate functions.

$$\mathcal{S}_1 := \left\{ (\rho_1, \rho_2) : \begin{array}{lll} \alpha_1 \in \mathcal{P}, & \sigma_1 \in \mathcal{K}, & \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{P}, & \sigma_2 \in \mathcal{K}, & \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \tag{21}$$

$$\mathcal{S}_2 := \left\{ (\rho_1, \rho_2) : \begin{array}{lll} \alpha_1 \in \mathcal{K}\backslash\mathcal{K}_\infty, & \sigma_1 \in \mathcal{K}, & \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{P}, & \sigma_2 \in \mathcal{K}, & \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \tag{22}$$

$$\mathcal{S}_3 := \left\{ (\rho_1, \rho_2) : \begin{array}{lll} \alpha_1 \in \mathcal{K}_\infty, & \sigma_1 \in \mathcal{K}, & \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{P}, & \sigma_2 \in \mathcal{K}, & \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \tag{23}$$

$$\mathcal{S}_4 := \left\{ (\rho_1, \rho_2) : \begin{array}{lll} \alpha_1 \in \mathcal{K}_\infty, & \sigma_1 \in \mathcal{K}, & \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{K}_\infty, & \sigma_2 \in \mathcal{K}, & \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \tag{24}$$

These sets have relationship of

$$\mathcal{S}_2 \subset \mathcal{S}_1, \quad \mathcal{S}_4 \subset \mathcal{S}_3 \subset \mathcal{S}_1, \quad \mathcal{S}_2 \cap \mathcal{S}_3 = \emptyset \tag{25}$$

which is illustrated by Fig.3. The implication of equation numbers in parentheses is given in this section later on.

A system theoretic explanation of the sets $\mathcal{S}_i$, $i = 1, ..., 4$ of supply rates can be given in terms of ISS and iISS properties[6], [7] for the system Σ_i satisfying Assumption 1. The system Σ_i is said to be iISS with respect to input (u_i, r_i) and state x_i if $\alpha_i \in \mathcal{P}$ and σ_i, $\sigma_{ri} \in \mathcal{K}$ hold. The function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ iISS Lyapunov function[7], [8]. If α_i is additionally a class $\mathcal{K}_\infty$ function, the system Σ_i is said to be ISS with respect to input (u_i, r_i) and state x_i, and the function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ ISS Lyapunov function[17]. The trajectory-based definition of ISS and iISS may be seen more often than the Lyapunov-based definition this paper adopts. The two types of definition are equivalent in the sense that the existence of ISS (iISS) Lyapunov functions is necessary and sufficient for ISS (iISS, respectively)[17], [7]. It is clear from the definition that ISS implies iISS. The converse is not true. The set $\mathcal{S}_4$ corresponds to interconnection of two ISS systems, while $\mathcal{S}_1$ corresponds to interconnection of two iISS systems. Supply rates belonging to $\mathcal{S}_3$ describe interconnection of an ISS system and an iISS system. The set $\mathcal{S}_2$ represents the set of supply rates with which the system Σ_1 may be ISS(see Remark 4), while Σ_2 is only supposed to be iISS. Note that, if the function α_i is not restricted to $\mathcal{P} \setminus \mathcal{K}$, there is a possibility that the system Σ_i is ISS. In contrast to $\mathcal{S}_3$, even in the case that Σ_1 is ISS, the situation of $(\rho_1, \rho_2) \in \mathcal{S}_2$ indicates that the function V_1 achieving Assumption 1 with $\alpha_1 \in \mathcal{K}_\infty$ is not available information.

Remark 4: It can be verified by the definition of ISS Lyapunov functions in [17] that a given triplet of $\{\alpha_i \in \mathcal{K}\backslash\mathcal{K}_\infty, \sigma_i \in \mathcal{K}, \sigma_{ri} \in \mathcal{P}_0\}$ accompanied by an iISS Lyapunov function satisfying

$$\limsup_{s\to\infty}\{\sigma_i(s) + \sigma_{ri}(s)\} \leq \lim_{s\to\infty} \alpha_i(s) \tag{26}$$

guarantees the system Σ_i to allow the existence of another triplet $\{\alpha_i \in \mathcal{K}_\infty, \sigma_i \in \mathcal{K}, \sigma_{ri} \in \mathcal{P}_0\}$ accompanied by another ISS Lyapunov function. Thus, even in the case of $\alpha_i \in \mathcal{K}\backslash\mathcal{K}_\infty$, the system Σ_i is ISS with respect to input (u_i, r_i) and state x_i if (26) holds.

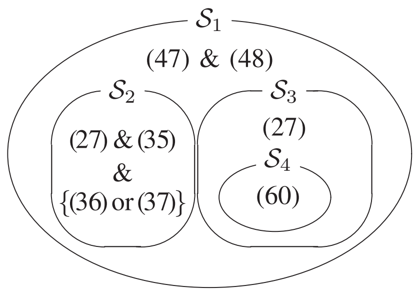

Fig. 3. Sets of supply rates and sufficient conditions for iISS and ISS

C. Feedback Connection of iISS and ISS Systems

We obtain the following theorem for interconnection of iISS and ISS systems.

Theorem 2: Suppose that supply rate functions $(\rho_1, \rho_2) \in \mathcal{S}_3$ are given. If there exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(w)]^k}{\sigma_1(w)} \le \frac{[\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^k}{\sigma_1(s)}, \quad \forall s\in\mathbb{R}_+ \qquad (27)$$

is satisfied, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x,r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \alpha_{cl}\in\mathcal{K},\ \sigma_{cl}\in\mathcal{P}_0\ . \qquad (28)$$

(ii) In the case of $\alpha_2 \in \mathcal{K}$, a solution to Problem 1 with respect to (28) is given by

$$\lambda_1(s) = \max_{w\in[0,s]} \nu c_1 c_2^q \underline{\delta}^{\frac{q}{q+1}} \frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(w)]^q}{\alpha_1\circ\bar{\alpha}_1^{-1}(w)} \qquad (29)$$

$$\lambda_2(s) = \nu q[\underline{\delta}^{\frac{1}{q+1}}\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^{q-1} \qquad (30)$$

where ν, $\underline{\delta}$ and q are any constants satisfying

$$\nu > 0, \quad 1 > \underline{\delta} > 0 \qquad (31)$$

$$c_2^q > [\underline{\delta}(c_1-1)]^{-1}, \quad q \ge k, \quad q > 1\ . \qquad (32)$$

(iii) In the case of $\alpha_2 \notin \mathcal{K}$, there exists $\hat{\alpha}_2 \in \mathcal{K}$ such that

$$\hat{\alpha}_2(s) \le \alpha_2(s) \qquad (33)$$

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(w)]^k}{\sigma_1(w)} \le \frac{[\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^k}{\sigma_1(s)}, \quad \forall s\in\mathbb{R}_+ \qquad (34)$$

hold, and a solution to Problem 1 is the same as *(ii)* except that α_2 is replaced by $\hat{\alpha}_2$.

It is stressed that there always exist ν, $\underline{\delta}$ and q fulfilling (31) and (32). It is required implicitly by (27) that the left hand side of (27) is finite at $w = 0$. It is due to the non-decreasing property of maximization and the fact that the right hand side of (27) takes finite value at all $s \in (0, \infty)$. The function $\lambda_1(s)$ given in (29) satisfies $\lim_{s\to 0^+}\lambda_1(s) < \infty$ since $\lim_{s\to 0^+}[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k/[\alpha_1\circ\bar{\alpha}_1^{-1}(s)] < \infty$ is implied by (27).

The following theorem explains that the condition (27) is still applicable to $\mathcal{S}_2$ covering a different type of supply rate when $\alpha_1^{-1}\circ c_1\sigma_1(s)$ makes sense.

Theorem 3: Suppose that supply rate functions $(\rho_1, \rho_2) \in \mathcal{S}_2$ are given. Assume that

$$\lim_{s\to\infty}\sigma_1(s) < \lim_{s\to\infty}\alpha_1(s) \qquad (35)$$

holds, and there exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that (27) is satisfied. Then, the statements (i), (ii) and (iii) in Theorem 2 are true if either of

$$c_1\limsup_{s\to\infty}\sigma_{r1}(s) < (1-\underline{\delta}^{\frac{1}{q+1}})(c_1-1)\lim_{s\to\infty}\alpha_1(s) \qquad (36)$$

$$\lim_{s\to\infty}\sigma_2(s) < \infty \qquad (37)$$

is satisfied.

The inverse α_1^{-1} denotes a function fulfilling $\alpha_1^{-1}\circ\alpha_1(s) = s$ for all $s \in \mathbb{R}_+$ although the domain of α_1^{-1} is not the entire $\mathbb{R}_+$. Note that $c_1\lim_{s\to\infty}\sigma_1(s) \le \lim_{s\to\infty}\alpha_1(s)$ is required by (27). The condition (35) ensures the existence of $c_1 > 1$ satisfying the requirement.

Calculations to check (27) can be made easier. The condition (27) can be replaced with either the pair of (38) and (39) or the pair of (43) and (39) in the following sense.

Lemma 1: Suppose that $(\rho_1, \rho_2) \in \mathcal{S}_2 \cup \mathcal{S}_3$.

(i) The pair of conditions

$$\frac{[\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^k}{\sigma_1\circ\underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \qquad (38)$$

$$c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \le \alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \quad \forall s\in\mathbb{R}_+ \qquad (39)$$

for a constant $k > 0$ implies (27). Conversely, if a constant $k > 0$ satisfies (27), there exists $\hat{\alpha}_2 \in \mathcal{K}$ satisfying

$$\hat{\alpha}_2(s) \le \alpha_2(s), \quad \forall s\in\mathbb{R}_+ \qquad (40)$$

$$\frac{[\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}(s)]^k}{\sigma_1\circ\underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \qquad (41)$$

$$c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \le \hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \quad \forall s\in\mathbb{R}_+\ . \qquad (42)$$

(ii) The pair of conditions

$$\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s)]^k}{\sigma_1(s)} \text{ is non-decreasing} \qquad (43)$$

and (39) for a constant $k > 0$ implies (27). Conversely, if a constant $k > 0$ satisfies (27), there exists $\hat{\sigma}_2 \in \mathcal{K}$ satisfying

$$\sigma_2(s) \le \hat{\sigma}_2(s), \quad \forall s\in\mathbb{R}_+ \qquad (44)$$

$$\frac{[\hat{\sigma}_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s)]^k}{\sigma_1(s)} \text{ is non-decreasing} \qquad (45)$$

$$c_2\hat{\sigma}_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \le \alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \quad \forall s\in\mathbb{R}_+\ . \qquad (46)$$

The property (35) implies that the system Σ_1 is ISS with respect to input u_1 and state x_1 although α_1 is not a class $\mathcal{K}_\infty$ function. For a system Σ_1 *without the exogenous signal* r_1, the condition (35) guarantees the existence of a $\mathbf{C}^1$ ISS Lyapunov function *neglecting* r_1 for another supply rate composed of a new pair $\{\alpha_1 \in \mathcal{K}_\infty,\ \sigma_1 \in \mathcal{K}\}$[17]. It should be stressed that the condition (35) does not ensure the ISS property of Σ_1 with respect to input r_1.

The criterion (27) is by no means applicable if α_1 does not belong to $\mathcal{K}$. When we know $\alpha_1 \in \mathcal{P}$ only, we need to consult the following theorem which provides a criterion of (47) and (48) accepting supply rates in the form of iISS.

Theorem 4: Suppose that supply rate functions $(\rho_1, \rho_2) \in \mathcal{S}_1$ are given. If there exist $c_i > 0$, $i = 1, 2$ and $k > 0$ such that

$$\left.\begin{array}{l}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^k \le c_1\alpha_1(\bar{\alpha}_1^{-1}(s)) \\ c_2\sigma_1(\underline{\alpha}_2^{-1}(s)) \le [\alpha_2(\bar{\alpha}_2^{-1}(s))]^k\end{array}\right\} \ \forall s \in \mathbb{R}_+ \quad (47)$$

$$c_1 < c_2 \quad (48)$$

are satisfied, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \ \alpha_{cl} \in \mathcal{P},\ \sigma_{cl} \in \mathcal{P}_0 \ . \quad (49)$$

(ii) In the case of $k \ge 1$ and $\alpha_2 \in \mathcal{K}$, a solution to Problem 1 with respect to (49) is given by

$$\lambda_1 = \frac{\nu c_1}{\delta^2}, \quad \lambda_2(s) = \nu k[\delta\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^{k-1} \quad (50)$$

where ν is any positive constant, and

$$\delta = \left(\frac{c_1}{c_2}\right)^{\frac{1}{k+2}} \ . \quad (51)$$

(iii) In the case of $k < 1$ and $\alpha_1 \in \mathcal{K}$, a solution to Problem 1 with respect to (49) is given by

$$\lambda_1(s) = \frac{\nu}{k}[\delta\alpha_1 \circ \bar{\alpha}_1^{-1}(s)]^{(1-k)/k}, \quad \lambda_2 = \frac{\nu}{\delta^2 c_2^{1/k}} \quad (52)$$

where ν is any positive constant, and

$$\delta = \left(\frac{c_1}{c_2}\right)^{\frac{1}{1+2k}} \ . \quad (53)$$

(iv) In the case of $\alpha_i \notin \mathcal{K}$, there exist $\hat{\alpha}_i \in \mathcal{K}$, $i = 1, 2$ such that

$$\hat{\alpha}_i(s) \le \alpha_i(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2 \quad (54)$$

$$\left.\begin{array}{l}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^k \le c_1\hat{\alpha}_1(\bar{\alpha}_1^{-1}(s)) \\ c_2\sigma_1(\underline{\alpha}_2^{-1}(s)) \le [\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^k\end{array}\right\} \ \forall s \in \mathbb{R}_+ \quad (55)$$

hold, and a solution to Problem 1 is the same as *(ii)* and *(iii)* except that α_i is replaced by $\hat{\alpha}_i$.

Theorem 4 has advantages over Theorem 2 and Theorem 3 in the following points.

- Theorem 4 does not require each system Σ_i to be ISS with respect to the external signal r_i.
- Theorem 4 is applicable directly to a given pair of $\alpha_1, \alpha_2 \in \mathcal{P} \setminus \mathcal{K}_\infty$.

The conditions in (47) necessitate $\liminf_{s\to\infty} \alpha_1(s) > 0$ and $\liminf_{s\to\infty} \alpha_2(s) > 0$ since σ_1 and σ_2 are class $\mathcal{K}$.

Remark 5: If both the two systems Σ_i, $i = 1, 2$ satisfy $r_i(t) \equiv 0$, the pair of (47)-(48) implies that at least one system Σ_i of Σ_1 and Σ_2 is ISS with respect to input u_i and state x_i. In other words, the pair of (47)-(48) yields the following property.

$$\limsup_{s\to\infty} \alpha_j(s) < \lim_{s\to\infty} \sigma_j(s) \ \Rightarrow \ \limsup_{s\to\infty} \alpha_i(s) > \lim_{s\to\infty} \sigma_i(s),\ i \ne j \ . \quad (56)$$

Therefore, by virtue of *(iv)* in Theorem 4, at least one of Σ_1 and Σ_2 needs to be ISS with respect to input u_i and state x_i *under the assumption of* $r_i(t) \equiv 0$ although $\alpha_i \in \mathcal{P} \setminus \mathcal{K}_\infty$ may hold. To see (56), consider the supply rate (20) with $\alpha_i \in \mathcal{P} \setminus \mathcal{K}_\infty$ for $i = 1, 2$. Two conditions in (47) lead to

$$\left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^k \le \frac{c_1\alpha_1(\bar{\alpha}_1^{-1}(s))}{c_2\sigma_1(\underline{\alpha}_2^{-1}(s))}, \quad \forall s \in \mathbb{R}_+ \setminus \{0\} \ .$$

From (48), we obtain

$$\liminf_{s\to\infty} \left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^k \le \limsup_{s\to\infty} \frac{\alpha_1(\bar{\alpha}_1^{-1}(s))}{\sigma_1(\underline{\alpha}_2^{-1}(s))} \quad (57)$$

for $k > 0$. Since limiting values of σ_1 and σ_2 toward ∞ are guaranteed to be finite by (47)-(48) and $\alpha_i \in \mathcal{P} \setminus \mathcal{K}_\infty$, the claim (56) follows. The requirement of (56) is natural in view of 'small gain' for the stability of the interconnection, and it can be intuitively explained as follows. Suppose that neither of the iISS systems Σ_1 and Σ_2 is ISS for $r_i(t) \equiv 0$. Then, there are no iISS Lyapunov functions whose supply rates satisfy $\alpha_i(\infty) \ge \sigma_i(\infty)$. Thus, in the absence of r_i, iISS Lyapunov functions $V_1(x_1)$ and $V_2(x_2)$ given *arbitrarily* satisfy

$$\frac{dV_i(x_i)}{dt} \le -\alpha_i(\bar{\alpha}_i^{-1}(V_i(x_i))) + \sigma_i(\underline{\alpha}_j^{-1}(V_j(x_j))) \quad (58)$$

for $i, j \in \{1, 2\}$, $i \ne j$ along the trajectories of Σ_i, and

$$\alpha_1(\infty) < \sigma_1(\infty), \quad \alpha_2(\infty) < \sigma_2(\infty) \ . \quad (59)$$

Due to (59), there exist sufficiently large $l_1, l_2 > 0$ such that $\alpha_1(\infty) < \sigma_1(\underline{\alpha}_2^{-1}(l_2))$ and $\alpha_2(\infty) < \sigma_2(\underline{\alpha}_1^{-1}(l_1))$ hold. We have $dV_i(x_i)/dt \ge 0$ for $x_i \in \mathbf{U}_i(l_i) = \{x_i \in \mathbb{R}^{n_i} : V_i(x_i) \ge l_i\}$ if we can assume that the pair $\{\alpha_i, \sigma_i\}$ is selected such that the gap in the inequality (58) is sufficiently small in $\mathbf{U}_i(l_i)$. Hence, the simultaneous property (59) contradicts the global asymptotic stability of $x = 0$.

Remark 6: According to Remark 4, even in the case that Theorem 2 does not cover an original triplet $\{\alpha_i, \sigma_i, \sigma_{ri}\}$ which Theorem 4 does, there may exist a transformed triplet $\{\alpha_i, \sigma_i, \sigma_{ri}\}$ accepted by Theorem 2. The process of transformation of supply rates, however, not only involves extra manipulation requiring users to have special knowledge, but also often causes conservatism in dissipation inequalities.

Combining theorems and a lemma in this subsection, we arrive at the following corollary which establishes iISS property of the feedback interconnection of iISS and ISS systems.

Corollary 1: The interconnected system Σ is iISS with respect to input r and state x if at least one of the following is satisfied.

(i) $(\rho_1, \rho_2) \in \mathcal{S}_1$. There exist $c_i > 0$, $i = 1, 2$ and $k > 0$ such that (47) and (48) hold.

(ii) $(\rho_1, \rho_2) \in \mathcal{S}_2$. The inequality (35) and one of (36) and (37) are satisfied. There exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that (39) and at least one of (38) and (43) hold.

(iii) $(\rho_1, \rho_2) \in \mathcal{S}_3$. There exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that (39) and at least one of (38) and (43) hold.

Remark 7: If we replace $|x_i|$ by $V_i(x_i)$ in the supply rate (20), the functions $\underline{\alpha}_i$ and $\bar{\alpha}_i$ disappear at all places in this section. The calculation to check the existence conditions is easier in this case. There is a trade-off between convenience in checking an existence condition and selecting a dissipative inequality. The use of $V_i(x_i)$ in the supply rate may not be appropriate in many cases.

Remark 8: The conditions (27), (39) and (47) are monotone in c_i, so that c_1 and c_2 can be obtained easily in a bisection manner if they exist. The computation of composite mappings is also straightforward. The evaluation of the inequality sign in (27), (39) and (47) by hand would be hard when complicated functions are involved. However, since every function is a scalar-valued continuous function of a single scalar variable, the inequality sign can be always checked very easily by numerical computation reliably. It is worth noting that λ_1 and λ_2 can be obtained explicitly in an analytical form even when the existence conditions are checked numerically. Analytical evaluation of the existence conditions is also possible in many cases with the help of standard computer software of symbolic manipulation. Finally, calculation of inverse functions by hand would be too complicated when difficult functions are chosen. Since every function needed to be inverted is an increasing scalar-valued continuous function of a single scalar variable, its inverse map obtained numerically is amenable to curve fitting tools for recovering analytically explicit expression with good approximation. In the case that one prefers a simple expression of an inverse function by hand, the function to be inverted can be always overbounded by another function yielding a simple inverse if one allows some conservatism.

D. Relation between Proposed Criteria and ISS Small-Gain Theorem

For interconnection consisting of ISS systems, solutions to the state-dependent scaling problem are obtained in the following form.

Theorem 5: Suppose that supply rate functions $(\rho_1, \rho_2) \in \mathcal{S}_4$ are given. If there exist $c_i > 1$, $i = 1, 2$ such that

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{60}$$

is satisfied, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \quad \alpha_{cl} \in \mathcal{K}_\infty,\ \sigma_{cl} \in \mathcal{P}_0 \ . \tag{61}$$

(ii) In the case of $\sigma_1 \in \mathcal{K}_\infty$, a solution to Problem 1 with respect to (61) is given by

$$\lambda_1(s) = \left[\nu_1 \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \times \left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right]^m \tag{62}$$

$$\lambda_2(s) = \frac{c_2}{\delta(c_2-1)} \left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right] \left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{m+1} \tag{63}$$

where $\nu_1 : s \in \mathbb{R}_+ \to \mathbb{R}_+$ is any non-decreasing continuous function satisfying

$$\nu_1(s) > 0, \quad \forall s \in (0, \infty) \tag{64}$$

and δ, τ_1 and m are any real numbers satisfying

$$0 \le m, \quad 0 < \delta < 1, \quad 1 < \tau_1 \le c_1 \tag{65}$$

$$\frac{\tau_1}{[\delta^2(\tau_1-1)(c_2-1)]^{\frac{1}{m+1}}} \le c_1 \ . \tag{66}$$

(iii) In the case of $\sigma_1 \notin \mathcal{K}_\infty$, there exists $\hat{\sigma}_1 \in \mathcal{K}_\infty$ such that

$$\sigma_1(s) \le \hat{\sigma}_1(s), \quad \forall s \in \mathbb{R}_+ \tag{67}$$

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{68}$$

hold, and a solution to Problem 1 is the same as *(ii)* except that σ_1 is replaced by $\hat{\sigma}_1$.

It is stressed that there always exist m, δ, τ_1 such that (65) and (66) hold. We can revisit the ISS small-gain theorem proposed in [2], [3] in view of Theorem 5 as follows.

Corollary 2: Assume that supply rate functions satisfy $(\rho_1, \rho_2) \in \mathcal{S}_4$. If there exist $c_i > 1$, $i = 1, 2$ such that (60) is satisfied, the interconnected system Σ is ISS with respect to input r and state x.

The ISS small-gain theorem is approached by this paper from the direction of the state-dependent scaling problem. Theorem 5 gives an explicit formula for a Lyapunov function establishing the ISS property of the feedback system. In fact, the Lyapunov function is (17) where λ_1 and λ_2 are given by (62) and (63), respectively. The ISS small-gain theorem proposed in [2], [3] is presented and proved originally by using trajectories of systems, and it has not provided formulas useful for constructing Lyapunov functions for design. Jiang et al.[19] focused on the equivalence between gain formulation and Lyapunov formulation rather than provide Lyapunov functions explicitly. The Lyapunov function leading to the ISS small-gain theorem is not necessarily unique, so that the existence of a smooth Lyapunov function in a different form is proved by [19]. This paper employs a formulation of Lyapunov functions which allow a smooth transition to stability criteria for systems more general than interconnection of ISS systems. The ISS small-gain theorem is recovered as a special case.

Remark 9: In ISS analysis of open-loop and cascade systems, Lyapunov functions have been used successfully by [6], [17], [18], [9]. This paper extends the use of the techniques to feedback systems, and rigorously demonstrates that Lyapunov functions in the form of [18], [9], [20] can be tailored for proving the ISS small-gain theorem.

We can expect that stability of interconnection of iISS systems should require more restrictive conditions than that of ISS systems. In fact, there are reasonable relationships between the ISS small-gain theorem and stability conditions developed for more general systems.

Theorem 6: *(i)* Assume $(\rho_1, \rho_2) \in \mathcal{S}_2 \cup \mathcal{S}_3$. If there exist $c_1 > 0$, $c_2 > 0$ and $k > 0$ such that (47)-(48) are satisfied, there exist also $c_1 > 1$, $c_2 > 1$ such that (27) holds.

(ii) Assume $(\rho_1, \rho_2) \in \mathcal{S}_4$. If there exist $c_1 > 1$, $c_2 > 1$ and $k > 0$ such that (27) holds, the inequality (60) is satisfied.

The broader the class of supply rate functions covered by a theorem is, the more restrictive the condition for the existence of a solution to the state-dependent problem(i.e., for stability) is. We can fill each section of the set of supply rate functions with a less restrictive condition as shown in Fig.3. Note that solutions to state-dependent scaling problems are not unique. For example, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 4 is a solution to the problem for supply rates considered in Theorem 2 and Theorem 5. In the same manner, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 2 is also a solution to Theorem 5.

E. Cascade of iISS and ISS Systems

As mentioned in Subsection III-A, solutions to Problem 1 are also able to establish stability of cascade connection of iISS and ISS systems in the configuration of Fig.2. In addition to (21)-(24), the sets

$$\mathcal{S}_2^\sharp := \left\{ (\rho_1, \rho_2) : \begin{array}{l} \alpha_1 \in \mathcal{P}, \qquad\quad \sigma_1 \in \mathcal{K},\, \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{K}\backslash\mathcal{K}_\infty, \sigma_2 \in \mathcal{K},\, \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \quad (69)$$

$$\mathcal{S}_3^\sharp := \left\{ (\rho_1, \rho_2) : \begin{array}{l} \alpha_1 \in \mathcal{P}, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{K}_\infty, \;\sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \quad (70)$$

of supply rate functions are considered in this subsection.

Corollary 3: The cascade system Σ_c is iISS with respect to input r and state x if one of the following is satisfied.

(i) $(\rho_1, \rho_2) \in \mathcal{S}_1$ holds. There exist $c_1 > 0$ and $k > 0$ such that

$$[\sigma_2(\underline{\alpha}_1^{-1}(s))]^k \le c_1 \alpha_1(\bar{\alpha}_1^{-1}(s)), \quad \forall s \in \mathbb{R}_+ \ . \quad (71)$$

(ii) $(\rho_1, \rho_2) \in \mathcal{S}_2$ holds. Either

$$\limsup_{s\to\infty} \sigma_{r1}(s) < \lim_{s\to\infty} \alpha_1(s) \quad (72)$$

or (37) is satisfied. There exists $k > 0$ such that

$$\lim_{s\to 0+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} < \infty \ . \quad (73)$$

(iii) $(\rho_1, \rho_2) \in \mathcal{S}_3$ holds. There exists $k > 0$ such that (73) holds.

(iv) $(\rho_1, \rho_2) \in \mathcal{S}_2^\sharp$ and $\lim_{s\to\infty} \sigma_2(s) < \lim_{s\to\infty} \alpha_2(s)$ hold.

(v) $(\rho_1, \rho_2) \in \mathcal{S}_3^\sharp$ holds.

The facts *(iv)* and *(v)* of Corollary 3 are natural extensions of a known fact that the cascade of an ISS system and a globally asymptotically stable system is globally asymptotically stable.

It is known that the cascade connection of ISS systems is ISS[6]. A Lyapunov-type proof can be found in [18], [9]. The same fact can be also extracted from Problem 1 as a special solution.

Corollary 4: The cascade system Σ_c is ISS with respect to input r and state x if $(\rho_1, \rho_2) \in \mathcal{S}_4$ holds.

IV. Extended Problem for Static Systems

This section discusses extention of the state-dependent scaling formulation and defines a second mathematical problem. Its importance and usefulness for establishing stability properties of interconnected systems involving static systems is demonstrated.

A. Problem Formulation

Consider the interconnected system Σ in Fig.1 again. In this section, we assume that Σ_1 is a static system described by

$$\Sigma_1 : \ z_1 = h_1(t, u_1, r_1) \ . \quad (74)$$

This system is connected to the dynamic system Σ_2 defined in (9) through $u_2 = z_1$. The state vector of the overall system Σ is $x = x_2 \in \mathbb{R}^n$, and $n = n_2$. It is assumed that $h_1(t, 0, 0) = 0$ holds for all $t \in [t_0, \infty)$, $t_0 \ge 0$. The function h_1 is assumed to be piecewise continuous in t, and locally Lipschitz in the other arguments. Instead of working directly with h_1, we assume we know a function $\rho_1(z_1, u_1, r_1)$ satisfying the following condition:

Assumption 2: The inequality

$$\rho_1(z_1, u_1, r_1) \ge 0, \quad \forall u_1 \in \mathbb{R}^{p_2}, r_1 \in \mathbb{R}^{m_1}, t \in \mathbb{R}_+ \quad (75)$$

is satisfied with a continuous function $\rho_1 : (z_1, u_1, r_1) \in \mathbb{R}^{p_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ satisfying $\rho_1(0, 0, 0) = 0$.

For convenience, we call ρ_1 a supply rate although energy is never stored by any static system. For the dynamic system Σ_2, we assume that Assumption 1 is satisfied.

The problem of establishing stability of the interconnection Σ comprising a dynamic system and a static system can be formulated into a mathematical problem called Problem 2.

Problem 2: Given a continuously differentiable function $V_2 : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and continuous functions $\rho_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ and $\rho_2 : (x_2, z_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{p_2} \times \mathbb{R}^{m_2} \to \mathbb{R}$, find continuous functions $\lambda_1 : (t, z_1, x_2, r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}^{p_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}_+$, $\lambda_2 : s \in \mathbb{R}_+ \to \mathbb{R}_+$, an increasing continuous function $\xi_1 : s \in [0, N] \to \mathbb{R}_+$ and a continuous function $\varphi_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}_+$ satisfying

$$\lambda_2(s) > 0 \quad \forall s \in (0, \infty) \quad (76)$$

$$\lim_{s\to 0^+} \lambda_2(s) < \infty \quad (77)$$

$$\int_1^\infty \lambda_2(s) ds = \infty \quad (78)$$

$$\xi_1(s) \ge 0 \quad \forall s \in [0, N] \quad (79)$$

$$\varphi_1(z_1, x_2, r_1) \ge 0, \ \forall z_1 \in \mathbb{R}^{p_2}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1} \quad (80)$$

such that

$$\begin{aligned} &\lambda_1(t, z_1, x_2, r_1, r_2) \left[-\xi_1(\varphi_1(z_1, x_2, r_1)) \right. \\ &\quad \left. + \xi_1(\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1)) \right] \\ &\qquad + \lambda_2(V_2(t, x_2)) \rho_2(x_2, z_1, r_2) \le \rho_e(x_2, r_1, r_2), \\ &\quad \forall z_1 \in \mathbb{R}^{p_2}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \end{aligned} \quad (81)$$

holds for some continuous function $\rho_e : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_2, 0, 0) < 0 \quad , \forall x \in \mathbb{R}^{n_2} \setminus \{0\} \quad (82)$$

where $N \in [0, \infty]$ is defined by

$$N = \sup_{(z_1, x_2, r_1) \in \mathbb{R}^{p_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1}} [\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1))] \quad (83)$$

Theorem 7: The equilibrium $x = 0$ of the interconnected system Σ is globally uniformly asymptotically stable for $r(t) \equiv 0$ if there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem

2. Furthermore, a $\mathbf{C}^1$ function $V_{cl} : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$defined with the solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ as

$$V_{cl}(t,x_2) = \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{84}$$

satisfies (18) for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$, and (19) holds along the trajectories of the system Σ.

Remark 1 is applicable to (82) and Theorem 7. When $\xi_1(s)$ is affine in s, the inequality (81) becomes

$$\lambda_1\xi_1(\rho_1) + \lambda_2(V_2)\rho_2 \le \rho_e \ . \tag{85}$$

The function φ_1 disappears from (81), so that, in the case of affine $\xi_1(s)$, a solution to Problem 2 becomes the triplet $\{\lambda_1, \lambda_2, \xi_1\}$. Problem 2 is milder than Problem 1. More precisely, Problem 1 has a solution only if Problem 2 is solved with $\xi_1(s) = s$ and $p_2 = n_1$ in the following sense.

Lemma 2: Suppose that a continuous function $\rho_e : (x_1,x_2,r_1,r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfies (16) and

$$\sup_{x_1\in\mathbb{R}^{n_1}} \rho_e(x_1,x_2,r_1,r_2) < \infty, \quad \forall x_2\in\mathbb{R}^{n_2}, r_i\in\mathbb{R}^{m_i} \tag{86}$$
$$\sup_{x_1\in\mathbb{R}^{n_1}} \rho_e(x_1,x_2,0,0) < 0, \quad \forall x_2 \in \mathbb{R}^{n_2}\setminus\{0\} \ . \tag{87}$$

Then, there exists a continuous function $\tilde{\rho}_e : (x_2,r_1,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ such that

$$\rho_e(x_1,x_2,r_1,r_2) \le \tilde{\rho}_e(x_2,r_1,r_2), \ \forall x_i,\in\mathbb{R}^{n_1}, r_i\in\mathbb{R}^{m_i} \tag{88}$$
$$\tilde{\rho}_e(x_2,0,0) < 0 \quad ,\forall x_2 \in \mathbb{R}^{n_2}\setminus\{0\} \ . \tag{89}$$

As explained in Remark 2, even in the presence of static systems, classical stability criteria such as the $\mathcal{L}_p$ small-gain theorem, the passivity theorems, criteria of Popov and circle types [1], [23], [24], [25] do not make use of the extension introduced by Problem 2. Indeed, due to Lemma 2, the classical criteria are explained by constant λ_i's and $\xi_1(s) = s$ which form a solution of Problem 1.

B. Interconnection of iISS and Static Systems

Consider supply rate functions in the form of

$$\rho_1(z_1,u_1,r_1) = -\alpha_i(|z_1|) + \sigma_i(|u_1|) + \sigma_{r1}(|r_1|) \tag{90}$$
$$\rho_2(x_2,u_2,r_2) = -\alpha_i(|x_2|) + \sigma_i(|u_2|) + \sigma_{r2}(|r_2|) \ . \tag{91}$$

It is supposed that α_i, σ_i, $\sigma_{ri} \in \mathcal{P}_0$ are known, but exact information of the differential equations (74) and (9) is not required. Define the following set of supply rate functions.

$$\mathcal{S}_5 := \left\{ (\rho_1,\rho_2) : \begin{array}{lll} \alpha_1 \in \mathcal{K}, & \sigma_1 \in \mathcal{K}, & \sigma_{r1} \in \mathcal{P}_0 \\ \alpha_2 \in \mathcal{P}, & \sigma_2 \in \mathcal{K}, & \sigma_{r2} \in \mathcal{P}_0 \end{array} \right\} \ . \tag{92}$$

This set satisfies

$$\mathcal{S}_2 \subset \mathcal{S}_5 \subset \mathcal{S}_1, \quad \mathcal{S}_4 \subset \mathcal{S}_3 \subset \mathcal{S}_5 \subset \mathcal{S}_1 \ . \tag{93}$$

Theorem 8: Suppose that supply rate functions $(\rho_1,\rho_2) \in \mathcal{S}_5$ are given. Assume that

$$\lim_{s\to\infty} \sigma_1(s) < \lim_{s\to\infty} \alpha_1(s) \ \text{ or } \ \alpha_1 \in \mathcal{K}_\infty \tag{94}$$

holds, and there exist $c_i > 1$, $i = 1,2$ such that

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{95}$$

is satisfied. If one of

$$c_1 \limsup_{s\to\infty} \sigma_{r1}(s) \le (c_1-1) \lim_{s\to\infty} \alpha_1(s) \ \text{ or } \ \alpha_1 \in \mathcal{K}_\infty \tag{96}$$
$$\lim_{s\to\infty} \sigma_2(s) < \infty \tag{97}$$

is satisfied, the following hold.

(i) Problem 2 is solvable with respect to a continuous function $\rho_e(x,r)$ of the form

$$\rho_e(x_2,r) = -\alpha_{cl}(|x_2|) + \sigma_{cl}(|r|)), \qquad \alpha_{cl}\in\mathcal{P}, \ \sigma_{cl}\in\mathcal{P}_0 \ . \tag{98}$$

Furthermore, $\alpha_{cl}\in\mathcal{K}_\infty$ is guaranteed if $\alpha_2\in\mathcal{K}_\infty$.

(ii) A solution to Problem 2 with respect to (98) is given by

$$\lambda_1 = \lambda_2 = \nu, \quad \varphi_1(s) = \alpha_1(s) \tag{99}$$
$$\xi_1(s) = \sigma_2 \circ \alpha_1^{-1}(s) \tag{100}$$

where ν is any positive constant. In the case of (97), the domain of ξ_1 is extended by

$$\xi_1(s) = s - h + \lim_{s\to\infty} \sigma_2(s) \ \text{ for } s \in [h,\infty) \tag{101}$$
$$h = \lim_{s\to\infty} \alpha_1(s) \ .$$

(iii) If $\sigma_{r1}(s) \equiv 0$ holds, the conditions (94) and (95) are replaced by

$$\lim_{s\to\infty} \sigma_1(s) \le \lim_{s\to\infty} \alpha_1(s) \ \text{ or } \ \alpha_1 \in \mathcal{K}_\infty \tag{102}$$
$$c_2\sigma_2 \circ \alpha_1^{-1} \circ \sigma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{103}$$

respectively.

The case of $(\rho_1,\rho_2) \in \mathcal{S}_1$ which is not covered by Theorem 8 can be dealt with by Theorem 4 due to the inclusive relation between Problem 1 and Problem 2, i.e., Lemma 2. Note that (47) is replaced by

$$[\sigma_2(s)]^k \le c_1\alpha_1(s), \ c_2\sigma_1(\underline{\alpha}_2^{-1}(s)) \le [\alpha_2(\bar{\alpha}_2^{-1}(s))]^k \tag{104}$$

since $V_1(z_1) = \underline{\alpha}_1(|z_1|) = \bar{\alpha}_1(|z_1|) = |z_1|$ can be used fictitiously for the static system Σ_1. In contrast, for $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$, it is no use relying on solutions to Problem 1 although Lemma 2 is valid. In fact, Theorem 8 not only covers $\mathcal{S}_5$ which is broader than $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$, but also provides less conservative conditions for the existence of solutions than Theorem 2, 3 and 5 in each case of $\mathcal{S}_3$, $\mathcal{S}_2$ and $\mathcal{S}_4$. Consequently, Theorem 8 indicates that stability of interconnected systems can be established by milder conditions in the presence of static systems. This fact is summarized by the following.

Corollary 5: Suppose that Σ_1 is static. The interconnected system Σ is iISS with respect to input r and state x_2 if one of the following is satisfied.

(i) $(\rho_1,\rho_2) \in \mathcal{S}_1$ holds. There exist $c_i > 0$, $i = 1,2$ and $k > 0$ such that (104) and (48) hold.

(ii) $(\rho_1,\rho_2) \in \mathcal{S}_5$ holds. The condition (94) and one of (96) and (97) are satisfied. There exist $c_i > 1$, $i = 1,2$ such that (95) holds.

Furthermore, if $\alpha_2 \in \mathcal{K}_\infty$, the interconnected system Σ is ISS.

Corollary 6: Suppose that Σ_1 is static. The cascade system Σ_c is iISS with respect to input r and state x_2 if one of the following is satisfied.

(i) $(\rho_1, \rho_2) \in \mathcal{S}_1$ holds. There exist $c_1 > 0$ and $k > 0$ such that

$$[\sigma_2(s)]^k \le c_1\alpha_1(s), \quad \forall s \in \mathbb{R}_+ \ . \tag{105}$$

(ii) $(\rho_1, \rho_2) \in \mathcal{S}_5$ holds. Either of

$$\limsup_{s\to\infty} \sigma_{r1}(s) \le \lim_{s\to\infty} \alpha_1(s) \quad \text{or} \quad \alpha_1 \in \mathcal{K}_\infty \tag{106}$$

and (97) is satisfied.

Furthermore, the cascade system Σ_c is ISS if $\alpha_2 \in \mathcal{K}_\infty$ is satisfied additionally.

Remark 10: For static systems, the inequality constraints in (94) and (106) can be assumed without loss of generality. Violation of the constraints implies that the functions α_1 σ_1 and σ_{r1} form a unreasonably loose bound for the system Σ_1. For instance, in the case of $r_1(t) \equiv 0$, if $\sigma_1(\infty) > \alpha_1(\infty)$ holds, Assumption 2 does not guarantee that finite $u_1(t)$ results in finite $z_1(t)$. This fact contradicts the assumption that $h_1(t, u_1, 0)$ is locally Lipschitz with respect to u_1 on $\mathbb{R}^{n_{u1}}$. Thus,

$$\lim_{s\to\infty} \sigma_1(s) \le \lim_{s\to\infty} \alpha_1(s)$$

is necessary for excluding the unreasonable case. In the same way, unless the inequality

$$\lim_{s\to\infty} \sigma_1(s) < \lim_{s\to\infty} \alpha_1(s)$$

holds, the static system $h_1(t, u_1, r_1)$ satisfying Assumption 2 is not guaranteed to be locally Lipschitz with respect to u_1 on $\mathbb{R}^{n_{u1}}$ however small $r_1 > 0$ is. When u_1 is absent, we need

$$\limsup_{s\to\infty} \sigma_{r1}(s) \le \lim_{s\to\infty} \alpha_1(s)$$

to ensure that the static system Σ_1 defined by Assumption 2 is locally Lipschitz in $r_1 \in \mathbb{R}^{n_{r1}}$.

Remark 11: When $\{\alpha_1 \in \mathcal{K} \setminus \mathcal{K}_\infty,\ \sigma_1 \in \mathcal{K},\ \sigma_{r1} \in \mathcal{P}_0\}$ achieves Assumption 2, there exists another triplet $\{\hat{\alpha}_1 \in \mathcal{K}_\infty,\ \hat{\sigma}_1 \in \mathcal{K},\ \hat{\sigma}_{r1} \in \mathcal{P}_0\}$ satisfying Assumption 2, provided that (26) holds. However, this transformation sometimes causes conservatism in evaluating (95).

V. Examples

This section illustrates the effectiveness of the state-dependent scaling approach through several simple examples of the interconnected system Σ in Fig.1. It is shown how scaling functions are obtained explicitly, and how they give Lyapunov functions establishing stability properties of Σ. The first two examples demonstrate that the state-dependence of scaling functions, i.e., 'non-linearly scaled combination' of supply rates is vital for dealing with nonlinearities which are not covered by classical stability criteria.

Example 1: Consider the following equations.

$$\Sigma_1 : \ \dot{x}_1 = -\left(\frac{x_1}{x_1+1}\right)^2 + 3\left(\frac{x_2}{x_2+1}\right)^2 \tag{107}$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{4x_2}{x_2+1} + \frac{2x_1}{x_1+1} + 6r_2 \tag{108}$$

$$x_1(0), x_2(0) \in \mathbb{R}_+, \ r_2(t) \in \mathbb{R}_+ \ .$$

The interconnected system Σ is defined for $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ and $r_2 \in \mathbb{R}_+$. Indeed, $x(0) \in \mathbb{R}_+^2$ and $r_2(t) \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$ imply that $x(t) \in \mathbb{R}_+^2, \forall t \in \mathbb{R}_+$. Although this example is for a compact illustration of theoretical development in this paper, it is motivated by models of biological processes which usually involve Monod nonlinearities and exhibit the non-negative property. The two systems are iISS with respect to input (u_i, r_i) and state x_i, where $u_1 = x_2$ and $u_2 = x_1$ hold, and r_1 is null. Neither Σ_1 nor Σ_2 is ISS with respect to input (u_i, r_i) and state x_i. Due to the non-negative property, the simplest choices of supply rates for Σ_1 and Σ_2 are $\rho_1 = \dot{x}_1$ and $\rho_2 = \dot{x}_2$ associated with iISS Lyapunov functions $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$. It is not difficult to calculate $\lambda_1(x_1)$ and $\lambda_2(x_2)$ achieving the scalar inequality (15) of Problem 1. The sum of scaled supply rates is

$$\begin{aligned} S(x, r_2) &= \lambda_1\rho_1 + \lambda_2\rho_2 \\ &= -\left[\lambda_1\left(\frac{x_1}{x_1+1}\right)^2 - 2\lambda_2\frac{x_1}{x_1+1}\right] \\ &\quad -\left[4\lambda_2\frac{x_2}{x_2+1} - 3\lambda_1\left(\frac{x_2}{x_2+1}\right)^2\right] + 6\lambda_2 r_2 \ . \end{aligned}$$

There are no constants $\lambda_1, \lambda_2 > 0$ which render $S(x, 0)$ negative definite. Thus, we need to introduce a function to at least one of λ_1 and λ_2. Since $\rho_1 = \dot{x}_1$ and $\rho_2 = \dot{x}_2$ yield

$$\alpha_1(s) = \left(\frac{s}{s+1}\right)^2, \ \sigma_1(s) = 3\left(\frac{s}{s+1}\right)^2$$
$$\alpha_2(s) = \frac{4s}{s+1}, \ \sigma_2(s) = \frac{2s}{s+1}, \ \sigma_{r2}(s) = 6s$$

which fulfill $(\rho_1, \rho_2) \in \mathcal{S}_1$, we use Theorem 4 to obtain $\{\lambda_1, \lambda_2\}$ satisfying (15). The inequalities in (47) are obtained as

$$2^k\left(\frac{s}{s+1}\right)^k \le c_1\left(\frac{s}{s+1}\right)^2, \quad \forall s \in \mathbb{R}_+$$
$$3c_2\left(\frac{s}{s+1}\right)^2 \le 4^k\left(\frac{s}{s+1}\right)^k, \quad \forall s \in \mathbb{R}_+ \ .$$

These two inequalities and $0 < c_1 < c_2$ are achieved by $k = 2$, $c_1 = 4$ and $c_2 \in (4, 16/3]$. Thus, the iISS property of Σ with respect to input r_2 and state x follows from Corollary 1*(i)*. Theorem 4 provides a solution to Problem 1 as

$$\lambda_1(s) = 1, \quad \lambda_2(s) = bs/(s+1), \ b \in [1.6119, 2) \ . \tag{109}$$

An iISS Lyapunov function of Σ is obtained from (17) as

$$V_{cl}(x) = x_1 + b(x_2 - \log(x_2 + 1)), \ b \in [1.6119, 2) \ .$$

The value of $\lambda_1\rho_1 + \lambda_2\rho_2$ with (109) and $b = 1.7$ is plotted on the state space in Fig.4. For visual simplicity, the surface is drawn for $r_2 = 0$. It is observed that the surface of $\lambda_1\rho_1 + \lambda_2\rho_2$ is below the horizontal plane of zero. This confirms that Problem 1 is solved by the choice (109),

Example 2: Consider

$$\Sigma_1 : \dot{x}_1 = -\frac{2x_1}{x_1+1} + \frac{x_2}{x_2+1}, \ x_1(0) \in \mathbb{R}_+ \tag{110}$$

$$\Sigma_2 : \dot{x}_2 = -\frac{4x_2}{x_2+1} + x_1, \quad x_2(0) \in \mathbb{R}_+ \ . \tag{111}$$

The interconnected system Σ satisfies $x \in \mathbb{R}_+^2$ or all $t \in \mathbb{R}_+$. The choice $V_1(x_1) = x_1$ yields $\dot{V}_1 = \rho_1(x_1, x_2)$ for

$$\alpha_1(s) = \frac{2s}{s+1},\ \sigma_1(s) = \frac{s}{s+1}\ .$$

Since (35) holds, the system Σ_1 is ISS with respect to input x_2 and state x_1. The system Σ_2 is not ISS since we have $x_2 \to \infty$ as $t \to \infty$ for $x_1(t) \equiv 5$. The system Σ_2 is iISS since the choice $V_2(x_2) = x_2$ gives $\dot{V}_2 = \rho_2(x_2, x_1)$ and

$$\alpha_2(s) = \frac{4s}{s+1},\ \sigma_2(s) = s\ .$$

It is easily seen that Problem 1 is not solvable by any constant λ_1 and λ_2. Due to $(\rho_1, \rho_2) \in \mathcal{S}_2$ and (35), Theorem 3 can be used for finding λ_1 and λ_2, From

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) = \frac{c_1c_2s}{(2-c_1)s+2}$$

it follows that (39) is identical with the pair of $8 - c_1c_2 - 4c_1 \geq 0$ and $c_1 \leq 2$. Since there exist such c_1, $c_2 > 1$, we have (39) fulfilled. In addition, we obtain (43) as well as (38) for $k = 1$ from

$$\frac{\sigma_2(s)}{\alpha_1(s)} = \frac{s+1}{2},\quad \frac{\alpha_2(s)}{\sigma_1(s)} = 4\ .$$

According to Corollary 1*(ii)*, the origin $x = 0$ is globally asymptotically stable. A solution

$$\lambda_1(x_1) = x_1(x_1+1),\quad \lambda_2(x_2) = \frac{7x_2}{x_2+1} \tag{112}$$

of Problem 1 is given by (29) and (30) with $c_1 = 1.2$, $c_2 = 2.5$, $q = 2$, $\nu = 2c_1^{-1}c_2^{-2/3}$ and $\underline{\delta} = 0.9044$. A Lyapunov function of Σ is given by (17) as

$$V_{cl}(x) = \frac{x_1^3}{3} + \frac{x_1^2}{2} + 7(x_2 - \log(x_2+1))\ .$$

Figure 5 illustrates that the choice (112) solves Problem 1. It is worth noting that the solution and the stability cannot be derived from Theorem 4. In fact, the first inequality in (47) is not achievable by any $k > 0$ since we have

$$[\sigma_2(s)]^k = s^k,\quad \alpha_1(s) = \frac{2s}{s+1}\ .$$

Example 3: Consider

$$\Sigma_1:\ \dot{x}_1 = -x_1 - x_1^3 + \gamma_1 x_2,\quad x_1(0) \in \mathbb{R} \tag{113}$$

$$\Sigma_2:\ \dot{x}_2 = -x_2 - x_2^3 + \gamma_2 x_1,\quad x_2(0) \in \mathbb{R} \tag{114}$$

and their ISS Lyapunov functions $V_i(x_i) = x_i^2/2$, $i = 1, 2$, considered in [26]. Supply rates

$$\dot{V}_i = -x_i^2 - x_i^4 + \gamma_i x_i x_j \leq -\frac{x_i^2}{2} - x_i^4 + \frac{\gamma_i^2 x_j^2}{2}$$

are obtained by using Young's inequality. The sum of scaled supply rates is

$$\begin{aligned} S(x) &= \lambda_1\rho_1 + \lambda_2\rho_2 \\ &= -\lambda_1 x_1^4 - \lambda_2 x_2^4 - (\lambda_1 - \gamma_2^2\lambda_2)\frac{x_1^2}{2} - (\lambda_2 - \gamma_1^2\lambda_1)\frac{x_2^2}{2}\ . \end{aligned}$$

It is straightforward that Problem 1 is solved by 'constants' λ_1 and λ_2 satisfying

$$0 < \lambda_1\gamma_1^2 \leq \lambda_2 \leq \lambda_1\gamma_2^{-2}\ . \tag{115}$$

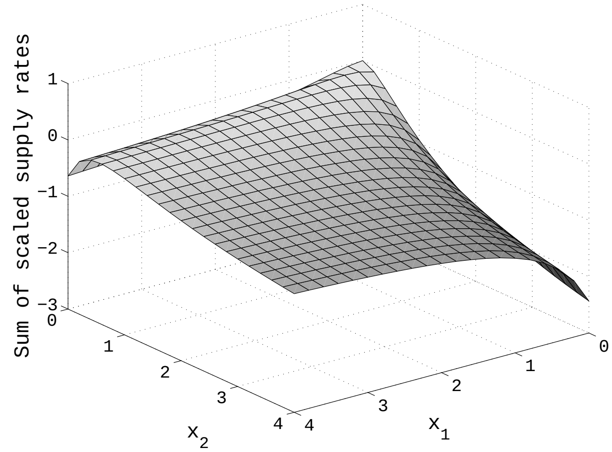

Fig. 4. State-dependently scaled combination of supply rates in Example 1.

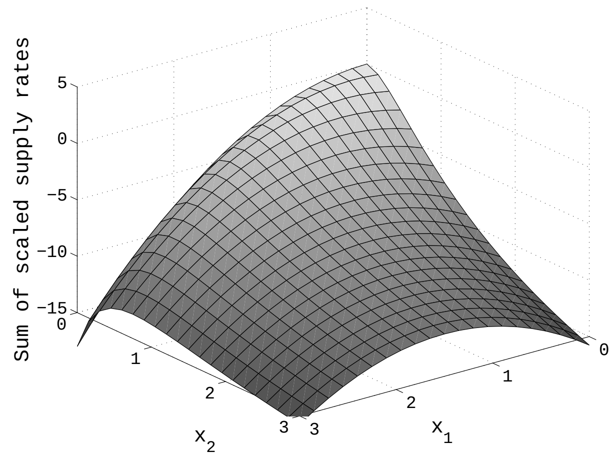

Fig. 5. State-dependently scaled combination of supply rates in Example 2.

Thus, the origin $x = 0$ of the interconnected system Σ is globally asymptotically stable for $|\gamma_1\gamma_2| \leq 1$. With 'any positive constants' λ_1 and λ_2, Theorem 1 also guarantees that, if $|\gamma_1\gamma_2| > 1$, there exist a bounded set toward which the state x converges. The set is easily determined by the smallest level set of $V_{cl} = \lambda_1 V_1 + \lambda_2 V_2$ including $S(x) \geq 0$. The question of 'global' asymptotic stability also fits in formulas presented in Subsection III-C. For example, we can apply even the most restricted formulas in Theorem 4 by considering $k = 1$, $\rho_e = -\lambda_1 x_1^4 - \lambda_2 x_2^4 + \tilde{\rho}_e$. and $\alpha_i(s) = s^4 + \tilde{\alpha}_i(s)$. Since $\tilde{\rho}_e$ is only required to be semi-negative definite for the global asymptotic stability, the equal sign is allowed in (48). The set of $c_1 \leq c_2$ and (47) is identical to $|\gamma_1\gamma_2| \leq 1$, and it ensures the 'global' asymptotic stability. The formulas in (50) give (115) at the same time.

VI. Conclusion

In this paper, stability criteria for interconnected iISS and ISS systems have been derived. In the course of the development, this paper has presented a new framework of state-dependent scaling for establishing stability properties of interconnected dissipative systems. State-dependent scaling problems are introduced as unified formulation without limitations on supply rates. The formulation can be considered as seamless incorporation of the dissipative approach[1], the integral-type of Lyapunov functions[18], [20] and the ISS small-gain technique[19]. If we restrict our attention to traditionally popular supply rates, classical stability criteria can be extracted as special cases where calculation of solutions to the state-dependent scaling problems is straightforward. The criteria are viewed as sufficient conditions for the existence of solutions. The solutions lead to Lyapunov functions of interconnected systems immediately. Unfortunately, it is not possible to find solutions of state-dependent scaling problems systematically for arbitrarily general systems. The existence of solutions is not trivial in general either. This fact motivated the author to deeply investigate the issues of how to obtain solutions and when they exist for systems which are not covered by stability criteria available previously. For this purpose, this paper has focused on interconnected systems consisting of iISS and ISS systems. As a result, explicit formulas for solutions have been derived. Sufficient conditions for the existence have been obtained as small-gain-like theorems which generalize the ISS small-gain theorem smoothly.

The developments of this paper have brought up some interesting issues. Further research is needed to pursue solutions to the state-dependent scaling problems for various types of supply rate. It is worth stressing that analytical computation is not the only way to make use of the developments of this paper. Using increasing power of computers and softwares, we can seek solutions numerically. While analytical investigation this paper mainly focuses on gives us guarantees of the effectiveness for some representative types of supply rate, numerical computation allows us to try to find solutions for more general supply rates. Computational methods in control theory have increasingly attracted public attention recently. Some examples are sum of squares programming[28], [29] and convex relaxation techniques which combine computational algebra and convex optimization. Problem 1 is jointly affine in the parameters λ_1 and λ_2, and Problem 2 is also jointly affine in these parameters. The affine property should be advantageous to numerical computation. It is an important and practical direction of future research to seek efficient numerical optimization algorithm for state-dependent scaling problems. Another direction of future research is to pursue analytical formulas for supply rates which are more general than the iISS property. This paper assumes that supply rates are given *a priori* for individual systems. For a practical example of the iISS property and manipulation to obtain functions of supply rates, the readers may refer to [8]. The issue of how to determine supply rates systematically in practice has not been answered satisfactorily in the literature yet. It needs to be addressed in the future.

APPENDIX

A. Proof of Theorem 1

The function $V_{cl}(t,x)$ defined by (17) is $\mathbf{C}^1$ since λ_1 and λ_2 are continuous. The assumptions of (10) and (12)-(14) imply the existence of $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$ satisfying (18). Due to (11), the time-derivative of $V_{cl}(t,x)$ along the trajectories of Σ satisfies

$$\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x_1} f_1 + \frac{\partial V_{cl}}{\partial x_2} f_2 \leq \lambda_1(V_1)\rho_1 + \lambda_2(V_2)\rho_2 \ .$$

From (15) we obtain (19). The property (16) guarantees global uniform asymptotic stability of $x = 0$ when $r(t) \equiv 0$.

B. Proof of Theorem 2

(i) and *(ii)* for $\alpha_2 \in \mathcal{K}$: Define δ and choose $\bar{\delta}$ as

$$\delta = \underline{\delta}^{\frac{1}{q+1}}, \quad \delta < \bar{\delta} < 1 \ . \tag{116}$$

The inequality (32) and $c_2 > 1$ ensure the existence of μ and $\tilde{\mu}$ satisfying $0 < \tilde{\mu} < \mu$ and

$$\left(\frac{c_2\tilde{\mu}}{\mu}\right)^q \geq \frac{1}{\underline{\delta}(c_1-1)} \ . \tag{117}$$

Suppose $\tau > 1$. Then, there exists $\tau_r > 1$ such that

$$1 - \frac{1}{\tau} - \frac{1}{\tau_r} \geq \bar{\delta}\left(1 - \frac{1}{\tau}\right) \tag{118}$$

is satisfied. Define $\theta_1 \in \mathcal{K}$ and $\theta_{r1} \in \mathcal{P}_0$ as follows:

$$\theta_1(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau\sigma_1(s), \quad \theta_{r1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_r\sigma_{r1}(s)$$

Combining calculations in individual cases separated by $\alpha_1(|x_1|) \geq \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) < \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) \geq \tau_r\sigma_{r1}(|r_1|)$ and $\alpha_1(|x_1|) < \tau_r\sigma_{r1}(|r_1|)$, we obtain

$$\begin{aligned}&\lambda_1(V_1(t,x_1))\left\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\right\} \\ &\quad\leq \bar{\delta}\left(-1+\frac{1}{\tau}\right)\lambda_1(V_1(t,x_1))\alpha_1(|x_1|) \\ &\qquad+\lambda_1(\theta_1(|x_2|))\sigma_1(|x_2|) + \lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)\end{aligned} \tag{119}$$

on the assumption that λ_1 is non-decreasing on $\mathbb{R}_+$. Using

$$\frac{1}{p} + \frac{1}{q} = 1 \ , \tag{120}$$

we define $p > 1$. The property $0 < \tilde{\mu} < \mu$ guarantees the existence of $\mu_r > 0$ satisfying

$$\frac{1}{\tilde{\mu}^p} \geq \frac{1}{\mu^p} + \frac{1}{\mu_r^p} \ . \tag{121}$$

Using Young's inequality, we obtain

$$\begin{aligned}&\lambda_2(V_2(t,x_2))\left\{-\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)\right\} \\ &\quad\leq -\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) \\ &\qquad+\frac{\nu q}{\tilde{\mu}^q}\left[\frac{1}{p}\left(\frac{\tilde{\mu}^q}{\nu q\mu}\lambda_2(V_2(t,x_2))\right)^p + \frac{\mu^q}{q}\sigma_2(|x_1|)^q\right. \\ &\qquad\left.+\frac{1}{p}\left(\frac{\tilde{\mu}^q}{\nu q\mu_r}\lambda_2(V_2(t,x_2))\right)^p + \frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q\right] \ .\end{aligned} \tag{122}$$

Define $\rho_e(x,r)$ with

$$\begin{aligned}\alpha_{cl}(s) = \min_{s=|x|}&\left\{(\bar{\delta}-\delta)\frac{\tau-1}{\tau}\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|)\right. \\ &\left.\quad+(1-\delta)\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right\} \\ \sigma_{cl}(s) = \max_{s=|x|}&\left\{\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)\right. \\ &\left.\quad+\nu\left(\frac{\mu_r}{\tilde{\mu}}\right)^q\sigma_{r2}(|r_2|)^q\right\} \ .\end{aligned}$$

The inequality (15) is achieved if the pair of λ_1 and λ_2 solves

$$-\delta\frac{\tau-1}{\tau}\lambda_1(s)\alpha_1(\bar{\alpha}_1^{-1}(s)) + \nu\left(\frac{\mu}{\tilde{\mu}}\right)^q[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq 0 \quad (123)$$

$$\frac{1}{p}\left(\frac{1}{\nu q}\right)^{p-1}\lambda_2(s)^p - \delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s)) + \lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0 \quad (124)$$

for all $s \in \mathbb{R}_+$ and if λ_1 is non-decreasing. Here, $\alpha_2 \in \mathcal{K}$ is assumed in obtaining (124). Substituting λ_2 given by (30) in (124), we obtain

$$\lambda_1(\theta_1(s))\sigma_1(s) \leq \nu[\delta\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q \ \forall s \in \mathbb{R}_+ \quad (125)$$

Hence, the pair of (123) and (124) holds if the non-decreasing function λ_1 given by (29) satisfies (125) and

$$\frac{\nu\mu^q\tau[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\tilde{\mu}^q\delta(\tau-1)\alpha_1(\bar{\alpha}_1^{-1}(s))} \leq \lambda_1(s), \quad \forall s \in \mathbb{R}_+ \ . \quad (126)$$

The choice of λ_1 satisfies (126) with $\tau = c_1$ by virtue of (116) and (117). It also satisfies (125) if

$$\max_{w\in[0,s]}\frac{[c_2\sigma_2(\underline{\alpha}_1^{-1}(\theta_1(w)))]^q}{\alpha_1(\bar{\alpha}_1^{-1}(\theta_1(w)))} \leq \frac{[\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{c_1\sigma_1(s)} \quad (127)$$

holds for all $s\in\mathbb{R}_+$ with $\tau = c_1$. Suppose (27) holds. Due to the maximization in (27) we have

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)$$

for all $s\in\mathbb{R}_+$. The increasing property of the left hand side of this inequality implies

$$\max_{w\in[0,s]} c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \ .$$

Combining this with (27), we obtain (127) for $q \geq k$. Thus, if (27) is satisfied, the non-decreasing functions λ_1 and λ_2 in (29) and (30) achieve (123) and (124) for $\tau = c_1$.
(i) and *(iii)* for $\alpha_2 \in \mathcal{P}\setminus\mathcal{K}$: The function $\hat{\alpha}_2 \in \mathcal{K}$ exists since the left hand side of (27) is non-decreasing and σ_1, $\bar{\alpha}_2^{-1} \circ \underline{\alpha}_2 \in \mathcal{K}$. The inequality (15) holds with α_2 if it holds with $\hat{\alpha}_2$.

C. Proof of Theorem 3

Case (36): The proof of Theorem 2 leads to the claim if τ_r and $\bar{\delta}$ are chosen carefully. The definition of θ_{r1} requires $\tau_r \limsup_{s\to\infty}\sigma_{r1}(s) \leq \lim_{s\to\infty}\alpha_1(s)$. The choice $\tau = c_1$ and (118) require τ_r to satisfy $\tau_r \geq c_1/(1-\bar{\delta})(c_1-1)$. Such τ_r and a number $\bar{\delta}$ satisfying $\delta < \bar{\delta} < 1$ exist if (36) holds.
Case (37): The definition (29) implies $\lambda_1 \in \mathcal{K}\setminus\mathcal{K}_\infty$, which guarantees $\sup_{t\in\mathbb{R}_+,\, x_1\in\mathbb{R}^{n_1}}\lambda_1(V_1(t,x_1)) \leq d$ for some finite $d > 0$. Remove τ_r and θ_{r1} and replacing $\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)$ with $d\sigma_{r1}(|r_1|)$ in (119).

D. Proof of Lemma 1

(i): Suppose that (38) and (39) hold for a constant $k > 0$. Since $\underline{\alpha}_2$ is class $\mathcal{K}$, we have

$$\frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s)]^k}{\sigma_1(s)} \leq \inf_{w\in[s,\infty)}\frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(w)]^k}{\sigma_1(w)} \ . \quad (128)$$

This implies (27). Conversely, we assume (27) for a constant $k > 0$. Let $\eta : s \in \mathbb{R}_+ \to \mathbb{R}_+$ denote the left hand side of (27). The function η is continuous and non-decreasing in $s \in \mathbb{R}_+$ due to the maximization, and it satisfies $\eta(s) > 0$ for all $s > 0$. Define a function $\hat{\alpha}_2 \in \mathcal{K}$ by

$$\hat{\alpha}_2(s) = [\eta \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)]^{1/k}[\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)]^{1/k} \ .$$

From (27) we obtain (40), (41) and (42).
(ii): On the assumption of (43), the condition (39) implies (27). Conversely, we assume (27) for a constant $k > 0$, which is equivalent to (128). Define $\eta : s \in \mathbb{R}_+ \to \mathbb{R}_+$ as the right hand side of (128). The function η is continuous and non-decreasing in s, and it satisfies $\eta(s) > 0$ for all $s > 0$. Let $d = \lim_{s\to\infty}\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s)$. Choose a function $\hat{\sigma}_2 \in \mathcal{K}$ so that

$$c_2\hat{\sigma}_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) = [\eta(s)]^{1/k}[\sigma_1(s)]^{1/k}$$

is satisfied for all $s \in \mathbb{R}_+$ and (44) holds for all $s \in [d,\infty)$. The existence is guaranteed since the right hand side is class $\mathcal{K}$. Then, we obtain (45) and (46). The condition (128) implies (44) for all $s \in [0,d)$.

E. Proof of Theorem 4

(i) and *(ii)* for $k = 1$: Using (47), we obtain $\lambda_1\rho_1 + \lambda_2\rho_2 \leq -(\lambda_1 - \lambda_2 c_1)\alpha_1(s) - (\lambda_2 - \lambda_1/c_2)\alpha_2(s)$. Let $\rho_e(x,r)$ be

$$\rho_e(x,r) = -(1-\delta)\left[\lambda_1\alpha_1(|x_1|) + \lambda_2\alpha_2(|x_2|)\right] + \lambda_1\sigma_{r1}(|r_1|) + \lambda_2\sigma_{r2}(|r_2|) \ .$$

The inequality (15) is achieved with (50) since the pair of (48) and (51) guarantees $c_1 \geq \delta c_1$ and $\delta^3 c_2 \geq c_1$.
(i) and *(ii)* for $k > 1$ and $\alpha_2 \in \mathcal{K}$: Set $q = k$. Let μ and $\tilde{\mu}$ be any positive constants satisfying

$$(\tilde{\mu}/\mu)^q = \delta \ . \quad (129)$$

Let $p > 1$ be defined by (120). Since (48) and (51) imply $0 < \delta < 1$, we have $0 < \tilde{\mu} < \mu$ which ensures the existence of $\mu_r > 0$ satisfying (121). Using Young's inequality, we obtain (122). Define

$$\alpha_{cl}(s) = \min_{s=|x|}(1-\delta)\left[\lambda_1\alpha_1(|x_1|) + \lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right]$$

$$\sigma_{cl}(s) = \max_{s=|x|}\left\{\lambda_1\sigma_{r1}(|r_1|) + \nu\left(\frac{\mu_r}{\tilde{\mu}}\right)^q\sigma_{r2}(|r_2|)^q\right\} \ .$$

The function $\rho_e(x,r)$ with $\lambda_1 > 0$ and $\lambda_2 \in \mathcal{K}$ given in (50) satisfies $\alpha_{cl} \in \mathcal{P}$ and $\sigma_{cl} \in \mathcal{K}$. Define λ_1 as in (50). A sufficient condition for (15) is obtained as

$$-\frac{\nu c_1}{\delta^2}\delta\alpha_1(|x_1|) + \nu\left(\frac{\mu}{\tilde{\mu}}\right)^q\sigma_2(|x_1|)^q \leq 0, \ \forall x_1\in\mathbb{R}^{n_1} \quad (130)$$

$$\left(\frac{1}{\nu q}\right)^{\frac{1}{q-1}}\frac{q-1}{q}\lambda_2(V_2(t,x_2))^{\frac{q}{q-1}} - \delta\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \frac{\nu c_1}{\delta^2}\sigma_1(|x_2|) \leq 0, \ \forall x_2\in\mathbb{R}^{n_2}, \ \forall t\in\mathbb{R}_+ \ . \quad (131)$$

Due to (129), the inequality (130) is identical to $[\sigma_2(s)]^q \leq c_1\alpha_1(s)$, which is implied by the first inequality in (47). Since

$\alpha_2 \in \mathcal{K}$ is non-decreasing, the inequality (131) holds if

$$\left(\frac{1}{\nu q}\right)^{\frac{1}{q-1}} \frac{q-1}{q}\lambda_2(s)^{\frac{q}{q-1}} - \delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s)) + \frac{\nu c_1}{\delta^2}\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0, \quad \forall s\in\mathbb{R}_+ \qquad (132)$$

is satisfied. This inequality (132) is equivalent to

$$\frac{\nu c_1}{\delta^2}\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq \nu\delta^q[\alpha_2(\bar{\alpha}_2^{-1}(s))]^q \quad \forall s \in \mathbb{R}_+$$

when $\lambda_2 \in \mathcal{K}$ is given by (50). Due to (51), the above inequality is identical to the second inequality in (47).
(i) and *(iii)* for $k < 1$ and $\alpha_1 \in \mathcal{K}$: Switch ρ_1 and ρ_2, and repeat the argument above with $k = 1/q$ by exchanging subscripts 1 and 2 each other.
(i) and *(iv)* for $\alpha_i \in \mathcal{P}\setminus\mathcal{K}$: The existence of $\hat{\alpha}_i \in \mathcal{K}$ satisfying follows from (47) and $\sigma_i, \underline{\alpha}_i^{-1}, \bar{\alpha}_i^{-1} \in \mathcal{K}$.

F. Proof of Theorem 5

(i) and *(ii)* for $\sigma_1 \in \mathcal{K}_\infty$: Let $\bar{\delta}$ be a real number satisfying $0 < \delta < \bar{\delta} < 1$, and set $\tau_2 = c_2$. For each $i = 1, 2$, there exists $\tau_{ri} > 1$ such that

$$1 - \frac{1}{\tau_i} - \frac{1}{\tau_{ri}} \geq \bar{\delta}\left(1 - \frac{1}{\tau_i}\right)$$

is satisfied since $\tau_i > 1$. Define $\theta_i \in \mathcal{K}$ and $\theta_{ri} \in \mathcal{P}_0$ as

$$\theta_i(s) = \bar{\alpha}_i \circ \alpha_i^{-1} \circ \tau_i\sigma_i(s), \quad \theta_{ri}(s) = \bar{\alpha}_i \circ \alpha_i^{-1} \circ \tau_{ri}\sigma_{ri}(s)$$

for $i = 1, 2$. Since the functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ $i = 1, 2$ given in (62) and (63) are non-decreasing, we obtain

$$\begin{aligned}&\lambda_i(V_i(t, x_i))\{-\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|)\} \\ &\leq \bar{\delta}\left(-1+\frac{1}{\tau_i}\right)\lambda_i(\underline{\alpha}_i(|x_i|))\alpha_i(|x_i|) \\ &\quad +\lambda_i(\theta_i(|u_i|))\sigma_i(|u_i|) + \lambda_i(\theta_{ri}(|r_i|))\sigma_{r1}(|r_i|) \qquad (133)\end{aligned}$$

for $i = 1, 2$. Thus, the inequality (15) is achieved if

$$\lambda_1(\theta_1(s))\sigma_1(s) \leq \delta\frac{\tau_2 - 1}{\tau_2}\lambda_2(\underline{\alpha}_2(s))\alpha_2(s), \ \forall s \in \mathbb{R}_+ \qquad (134)$$

$$\lambda_2(\theta_2(s))\sigma_2(s) \leq \delta\frac{\tau_1 - 1}{\tau_1}\lambda_1(\underline{\alpha}_1(s))\alpha_1(s), \ \forall s \in \mathbb{R}_+ \qquad (135)$$

hold. In fact, $\alpha_{cl} \in \mathcal{K}_\infty$ and $\sigma_{cl} \in \mathcal{P}_0$ in (61) are

$$\alpha_{cl}(s) = \min_{s=|x|}(\bar{\delta} - \delta)\sum_{i=1}^{2}\frac{\tau_i - 1}{\tau_i}\lambda_i(\underline{\alpha}_i(|x_i|))\alpha_i(|x_i|)$$

$$\sigma_{cl}(s) = \max_{s=|r|}\sum_{i=1}^{2}\lambda_i(\theta_{ri}(|r_i|))\sigma_{ri}(|r_i|) \ .$$

It is seen that (134) and (135) are fulfilled if λ_1 and λ_2 achieve

$$\begin{aligned}&\frac{\delta^2(\tau_1 - 1)(\tau_2 - 1)}{\tau_1\tau_2}\alpha_2(\underline{\alpha}_2^{-1}(\theta_2(s)))\alpha_1(s)\lambda_1(\underline{\alpha}_1(s)) \\ &\qquad \geq \sigma_2(s)\sigma_1(\underline{\alpha}_2^{-1}(\theta_2(s)))\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(\theta_2(s)))) \qquad (136)\end{aligned}$$

$$\lambda_1(\theta_1(s))\sigma_1(s) = \delta\frac{\tau_2 - 1}{\tau_2}\lambda_2(\underline{\alpha}_2(s))\alpha_2(s) \qquad (137)$$

for all $s \in \mathbb{R}_+$. Here, $\sigma_2 \in \mathcal{K}_\infty$ is assumed in obtaining (136). From $s \leq \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)$ it follows that $\tau_2\sigma_2(s) \leq \alpha_2(\underline{\alpha}_2^{-1}(\theta_2(s)))$. Thus, (136) is implied by

$$\begin{aligned}&\sigma_1(\underline{\alpha}_2^{-1}(\theta_2(s)))\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(\theta_2(s)))) \\ &\qquad \leq \frac{\delta^2(\tau_1 - 1)(\tau_2 - 1)}{\tau_1}\alpha_1(s)\lambda_1(\underline{\alpha}_1(s)) \ . \qquad (138)\end{aligned}$$

Remember that ν_1 and α_2 are non-decreasing. Using (62) and $\bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \leq s$, we can verify that the inequality (138) is satisfied if

$$\tau_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \qquad (139)$$

$$\frac{\tau_1}{[\delta^2(\tau_1-1)(\tau_2-1)]^{\frac{1}{m+1}}}\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \qquad (140)$$

hold. Since $\tau_1 \leq c_1$ and $\tau_2 \leq c_2$ hold. the inequality (60) guarantees (139). Due to (66), the inequality (60) also implies (140). On the other hand, using

$$\lambda_1 \circ \bar{\alpha}_1 \circ \alpha_1^{-1}(\tau_1 s) = \nu_1(s)\left[\alpha_2 \circ \sigma_1^{-1}(s)\right]s^m$$

$$\theta_1 \circ \underline{\alpha}_2^{-1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1\sigma_1 \circ \underline{\alpha}_2^{-1}(s)$$

we can verify that $\lambda_2(s)$ given in (63) solves (137). Hence, the inequality (15) is achieved by λ_1 and λ_2 given in (62) and (63). In the case of $\sigma_2 \in \mathcal{K} \setminus \mathcal{K}_\infty$, there is $\hat{\sigma}_2 \in \mathcal{K}_\infty$ such that $\sigma_2(s) \leq \hat{\sigma}_2(s)$ and

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\hat{\sigma}_2(s) \leq s$$

hold for all $s \in \mathbb{R}_+$. We can repeat the argument below (136) with $\hat{\sigma}_2$. Note that (62) and (63) do not involve σ_2.
(i) and *(iii)* for $\sigma_1 \in \mathcal{K} \setminus \mathcal{K}_\infty$: The function $\hat{\sigma}_1 \in \mathcal{K}_\infty$ exists since the right hand side of (60) is class $\mathcal{K}_\infty$.

G. Proof of Theorem 6

(i): Suppose that (47)-(48) holds for some $c_1 > 0$, $c_2 > 0$ and $k > 0$. Then, it follows that

$$\max_{w\in[0,s]}\frac{[\hat{c}_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \hat{c}_1\sigma_1(w)]^k}{\sigma_1(w)} \leq \hat{c}_1\hat{c}_2^k c_1$$

$$c_2 \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)}$$

hold for all $s \in \mathbb{R}_+$ with arbitrary $\hat{c}_1$, $\hat{c}_2 > 0$. Let $\hat{c}_2 = (c_2/c_1\hat{c}_1)^{1/k}$. On the assumption (48), there exists $\hat{c}_1 > 1$ such that $\hat{c}_2 > 1$ holds. Thus, we arrive at (27).
(ii): Due to (27), we obtain (60) from

$$\frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s)]^k}{\sigma_1(s)} \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)} \ .$$

H. Proof of Corollary 3

(i): The claim is obtained from Theorem 4 immediately.
(ii): The proof is different from the case of *(iii)* in the following points. The number $\underline{\delta} > 0$ in (32) can be made arbitrarily small by choosing sufficiently large $c_1, c_2 > 1$. Thus, we obtain (72) from (36) and sufficiently large c_1.
(iii): The proof is the same as that of Theorem 2 up to (126) and (125). The inequality (125) is satisfied automatically due

to $\sigma_1 = 0$. The condition (73) guarantees that the choice (29) of λ_1 solving (126) is non-decreasing.

(iv) and *(v)*: Exchanging subscripts 1 and 2 with each other, we consider the configuration in which Σ_1 is driven by Σ_2. Since u_2 is absent, we can choose an arbitrarily small $\sigma_2 \in \mathcal{K} \setminus \mathcal{K}_\infty$ fictitiously so that (27) is satisfied. Then, the proofs of Theorem 3 and Theorem 2 prove the claims.

I. Proof of Theorem 7

The increasing property of ξ_1 and the inequalities (75) and (80) yield $\xi_1(\varphi_1) \leq \xi_1(\varphi_1 + \rho_1)$. For V_{cl} given in (84), the existence of $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$ in (18) is guaranteed by (10) and (76)-(78). The time-derivative of V_{cl} satisfies

$$\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x} f_2 \leq \lambda_1(t, z_1, x_2, r)\left[-\xi_1(\varphi_1) + \xi_1(\varphi_1 + \rho_1)\right] + \lambda_2(V_2(t, x_2))\rho_2$$

along the trajectories of Σ since the range of λ_1 is in $\mathbb{R}_+$ and V_2 satisfies (11). The inequality (19) follows from (81). The property (82) implies that $x_2 = 0$ is globally uniformly asymptotically stable when $r(t) \equiv 0$.

J. Proof of Theorem 8

(i) and *(ii)*: Define δ and ζ as $0 < \delta = 1/c_2 < 1$ and $0 < \zeta = 1/(c_1 - 1)$. Then, we obtain

$$\begin{aligned} \sigma_2(|u_2|) &\leq \xi_1\left(\sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\right) \\ &\leq \sigma_2 \circ \alpha_1^{-1} \circ (1+1/\zeta)\sigma_1(|x_2|) \\ &\qquad + \xi_1 \circ (\zeta+1)\sigma_{r1}(|r_1|) \end{aligned} \quad (141)$$

from (75) and (92). Pick

$$\begin{aligned} \rho_e(x_2, r) = & -\nu(1-\delta)\alpha_2(|x_2|) \\ & + \nu\xi_1 \circ (\zeta+1)\sigma_{r1}(|r_1|) + \nu\sigma_{r2}(|r_2|) \ . \end{aligned}$$

The inequality (81) is satisfied if (95) holds. If $\alpha_1 \in \mathcal{K} \setminus \mathcal{K}_\infty$ and $\sigma_2(\infty) = \infty$ hold, we need $(\zeta+1)\limsup_{s\to\infty} \sigma_{r1}(s) \leq \alpha_1(\infty)$ in (141), which is the first inequality in (96).

(iii): The term of σ_{r1} disappears from (141). Thus, we can replace $(1+1/\zeta)$ with 1.

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