Necessary and Sufficient Small Gain Conditions for Integral Input-to-State Stable Systems: A Lyapunov Perspective

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***Abstract*—This paper is concerned with conditions for the stability of interconnected nonlinear systems consisting of integral input-to-state stable(iISS) systems with external inputs. The treatment of iISS and input-to-state stable (ISS) systems is unified. Both necessary conditions and sufficient conditions are investigated using a Lyapunov formulation. In the presence of model uncertainty, this paper proves that, for the stability of the interconnected system, at least one subsystem is necessarily ISS which is a stronger stability property in the set of iISS. The necessity of a small-gain-type property is also demonstrated. This paper proposes a common form of smooth Lyapunov functions which can establish the iISS and the ISS of the interconnection comprising iISS and ISS subsystems whenever the small-gain-type condition is satisfied. The result covers situations more general than the earlier study and removes technical conditions assumed in the previous literature. Global asymptotic stability is discussed as a special case.**

***Index Terms*—Nonlinear interconnected systems, Integral input-to-state stability, Small gain condition, Necessary condition, Lyapunov function.**

I. Introduction

The problem of establishing stability of interconnection has always been a fundamental issue within the systems and control community. As the nonlinearities we are dealing with are growing more complex, it becomes more difficult to derive conditions under which interconnected systems are stable. In order to succeed in analyzing and designing a wide range of nonlinear systems, we need stability criteria which are not only universal, but also accommodate nonlinearities effectively. For about a decade, the input-to-state stability (ISS) property has been a useful way to characterize nonlinearities in view of stability [22]. It did not take a long time for the ISS framework to be appreciated by the emergence of the ISS small-gain theorem [14], [15], [25]. The ISS small-gain theorem provides a sufficient condition for the stability of a feedback system comprised of ISS subsystems. The ISS small-gain theorem makes use of the idea of nonlinear loop gain, which is also implemented as a slightly different differential inequality in [19]. There is another important class of systems which are not necessarily ISS. This is characterized by the integral input-to-state stability (iISS) property [2], [23]. Those systems have finite nonlinear gain only in a very weak sense(See [13] for an illustrative example). In contrast to ISS systems, because of the weakness of gain, the cascade of iISS systems are not always stable [3], [4]. In spite of such a weak gain property, a stability criterion covering iISS systems in feedback configurations has been developed recently by one of the authors [10], which is a result of the Lyapunov constructive approach presented in [11]. The criterion gives a sufficient condition for iISS property of interconnected iISS systems in the form of a small-gain property. The possibility of establishing stability for the feedback interconnection of iISS systems by means of gain conditions is followed up by a nullcline approach [1] in the absence of external signals. Generalizing the proposed result of [1] to the case of external stability with respect to external signals is by no means easy. As a matter of fact, the relationship between the nullcline approach and the Lyapunov constructive approach has not been investigated yet.

The purpose of this paper is mainly threefold. One is to derive necessary conditions for the stability of interconnected systems in order to show how reasonable small-gain-type criteria are. Another is to unify the treatment of iISS and ISS systems by merging the two types of small-gain conditions derived from the two types of Lyapunov functions dealing with iISS and ISS separately. The third objective is to provide Lyapunov functions in the situations considered by the nullcline approach to global asymptotic stability(GAS) in the absence of external signals. We place an emphasis on Lyapunov functions to accomplish all the points. The previous work [11] employs nonidentical Lyapunov functions for the ISS case and the iISS case. This paper develops a single unified formula applicable equally to iISS systems and ISS systems. The condition of a small-gain type proposed in [11] for iISS systems looks more complicated and more restrictive than the small-gain condition for ISS systems. This paper not only merges the two small-gain-type conditions, but also removes the assumption of uniform contraction used in [11]. The removal of the uniformity assumption allows the nonlinear loop gain to approach unity asymptotically as the magnitude of signals tends to zero and infinity. The unification and the generalization of Lyapunov functions also enable us to come to the point where the necessity of the small-gain condition holds. Furthermore, it is shown that at least one of the subsystems in the loop needs to be ISS with respect to feedback input.

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The necessity of stability criteria has been an important issue in the area of robust control, e.g., [6]–[8], [26]. Very often, it is hard to obtain a mathematical model which completely captures dynamics of a physical system. Then, there is a mismatch between the model and the reality which is widely known under the name of uncertainty. The common idea to deal with the uncertainty is to model the system as belonging to a set. Stability margins characterize how modeling errors might affect the stability of a system by measuring how close the system is to instability. A stability criterion can provide a precise measure of stability robustness only if the criterion is proved to be necessary as well as sufficient for a set describing the uncertain system. For nonlinear systems, in a recent paper [1], the tightness of a nonlinear small-gain condition for stability without external signals is discussed by means of examples. To the best of the authors' knowledge, the necessity of small-gain conditions has not been investigated in view of Lyapunov functions for interconnections consisting of general ISS and iISS subsystems with external signals. In the framework of the input-output theory initiated by Zames and Sandberg in the 1960s [20], [27], it is worth mentioning that there have been several important studies on the necessity of small-gain conditions involving only the linear gains, i.e., $\mathcal{L}_p$-type gain [9], [21]. Those results have been generalized by using dissipative inequalities in an integral form [5], where the notion of conditional gain is introduced to render their new small-gain property necessary for the input-output stability of the interconnection made of a nominal system and an uncertainty. To address the issue of the necessity of small-gain conditions, this paper not only brings up a weak nonlinear gain property of iISS, but also develops a new approach based on Lyapunov functions to obtain GAS of the equilibrium of interest as well as the stability with respect to exogenous disturbances. Furthermore, this paper compares the necessity fact for nonlinear systems with that for linear systems, and highlights important differences caused by the diversity of nonlinearity.

The following notations are used. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Negation is $\neg$. The interval $[0,\infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. The Euclidean norm of a vector in $\mathbb{R}^n$ is denoted by $|\cdot|$. The identity map on $\mathbb{R}$ is denoted by $\mathbf{Id}$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be of class $\mathcal{K}$ and written as $\gamma\in\mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0)=0$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ and written as $\gamma\in\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty}\gamma(r)=\infty$. We write $\gamma\in\mathcal{P}_0$ for a function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ if it is a continuous function satisfying $\gamma(0)=0$. The set of $\gamma\in\mathcal{P}_0$ satisfying $\gamma(s)>0$ for all $s\in\mathbb{R}_+\setminus\{0\}$ is denoted by $\gamma\in\mathcal{P}$. For a function $h\in\mathcal{P}$, we write $h\in\mathcal{O}(>L)$ with a non-negative number L if there exists a positive number $K>L$ such that $\limsup_{s\to 0+}h(s)/s^K<\infty$ holds. We write $h\in\mathcal{O}(L)$ when $K=L$. As for limiting value of functions $f,g:\mathbb{R}_+\to\mathbb{R}_+$, we use the simple notation $\lim f(s)=\lim g(s)$ to describe $\{\lim f(s)=\infty \ \wedge\ \lim g(s)=\infty\}\ \vee\ \{\infty>\lim f(s)=\lim g(s)\}$. In a similar manner, $\lim f(s)\geq\lim g(s)$ denotes $\{\lim f(s)=\infty\ \vee\ \infty>\lim f(s)\geq\lim g(s)\}$. A system $\dot{x}=f(x,r)$ is said to be 0-GAS if the 0-input system $\dot{x}=f(x,0)$ has a unique equilibrium which is globally asymptotically stable. Due to space limitation and for want of readability, several proofs are omitted but are available from the authors upon request [12].

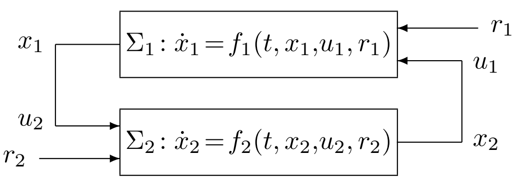


Fig. 1. Interconnected system Σ.

II. Motivational Examples

This section illustrates the main features of this paper using two examples. Consider the interconnected system Σ shown in Fig.1, and suppose that the systems Σ_1 and Σ_2 satisfy the following differential dissipation inequalities:

$$\Sigma_i \ :\ \dot{S}_i(|x_i|)\leq -\alpha_i(|x_i|)+\sigma_i(|x_{3-i}|)+\sigma_{ri}(|r_i|),\quad i=1,2, \tag{1}$$

where the supply rates are given by

$$\alpha_1(s)=\frac{\beta s^2}{s^2+\beta},\ \sigma_1(s)=s^2, \alpha_2(s)=s^4,\ \sigma_2(s)=\left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^2 \tag{2}$$

for some $\beta>0$, $\gamma>0$, $\sigma_{ri}\in\mathcal{P}_0$ and some storage function $S_i\in\mathcal{K}_\infty$, $i=1,2$. Note that, for each fixed s, as $\beta\to\infty$, the above functions in (2) converge to

$$\alpha_1(s)=s^2,\ \sigma_1(s)=s^2,\quad \alpha_2(s)=s^4,\ \sigma_2(s)=\gamma^2 s^4 .$$

For simplicity, these supply rates are referred to as $\beta=\infty$. It is directly verified that

$$\alpha_1^{-1}\circ c_1\sigma_1\circ\alpha_2^{-1}\circ c_2\sigma_2(s)\leq s,\quad \forall s\in\mathbb{R}_+ \tag{3}$$

is equivalent to $c_1\sqrt{c_2}\leq 1/\gamma$ for each $\beta\in(0,\infty]$. Thus, there exist c_1, $c_2>1$ such that (3) holds if and only if $\gamma<1$. This paper will show that, for an arbitrary pair $\sigma_{r1},\sigma_{r2}\in\mathcal{P}_0$, the function

$$V_{cl}(x)=\int_0^{S_1(|x_1|)}\lambda_1(s)ds+\int_0^{S_2(|x_2|)}\lambda_2(s)ds \tag{4}$$

establishes iISS of the interconnected system Σ with respect to input (r_1,r_2) and state (x_1,x_2) if $\gamma<1$. The integrands $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ derived by this paper are

$$\lambda_1(s)=\frac{d(c_2-1)}{c_2}\left(\frac{\beta(S_1^{-1}(s))^2}{c_1((S_1^{-1}(s))^2+\beta)}\right)^{K+1} \tag{5}$$

$$\lambda_2(s)=(S_2^{-1}(s))^{2K} \tag{6}$$

$$c_1=\frac{1}{2}\left(1+\frac{1}{\sqrt{c_2}\gamma}\right),\ c_2=\frac{1}{4}\left(1+\frac{1}{\gamma}\right)^2,\ d=\sqrt{\gamma c_1\sqrt{c_2}} \tag{7}$$

$$q=\frac{\gamma c_1\sqrt{c_2}}{1-\gamma c_1\sqrt{c_2}}\left(\frac{1}{(c_1-1)(c_2-1)}-1\right) \tag{8}$$

$$K=\begin{cases} 0 & ,\ \text{if } (c_1-1)(c_2-1)>1 \\ \max\{1,q\} & ,\ \text{otherwise}\end{cases} . \tag{9}$$

Note that, in the case of $\beta < \infty$, the system Σ_1 is not necessarily ISS with respect to input x_2 and state x_1, but it is always iISS. However, Σ_1 becomes ISS if $\beta = \infty$. This paper demonstrates that iISS systems and ISS systems can be dealt with equally by a common small-gain condition (3) and a common formula for constructing a Lyapunov function of the feedback loop. In fact, the function V_{cl} with functions λ_1, λ_2 defined as in (5)-(9) serves as an iISS Lyapunov function for each $\beta \in (0, \infty]$ whenever $\gamma < 1$ holds. In the case of $\beta = \infty$, it becomes an ISS Lyapunov function. This paper will also show that there is always a counterexample when $\gamma \geq 1$ holds in each case of $\beta \in (0, \infty]$. Let

$$\Sigma_1: \quad \dot{x}_1 = \frac{8}{3}\left\{\frac{-\beta|x_1|^{\frac{1}{2}}}{|x_1|^2+\beta} + \left(\frac{\beta^3(\tilde{\sigma}_1(|u_1|)+\tilde{\sigma}_{r1}(|r_1|))}{(|x_1|^2+\beta)^3}\right)^{\frac{1}{4}}\right\}x_1, \ x_1 \in \mathbb{R}^2 \tag{10}$$

$$\tilde{\sigma}_1(s) = \begin{cases} 100s^4 & , \quad 0 \leq s < 0.1 \\ s^2 & , \quad 0.1 \leq s \end{cases}$$

$$\Sigma_2: \quad \dot{x}_2 = \frac{8}{3}\left\{-|x_2|^{\frac{5}{2}} + |x_2|^{\frac{3}{2}}(\tilde{\sigma}_2(|u_2|) + \tilde{\sigma}_{r2}(|r_2|)))^{\frac{1}{4}}\right\}x_2, \quad x_2 \in \mathbb{R}^2 \tag{11}$$

$$\tilde{\sigma}_2(s) = \begin{cases} \left(\frac{0.01+\beta}{0.01\gamma\beta}\right)^2 \left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^4 & , \quad 0 \leq s < 0.1 \\ \left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^2 & , \quad 0.1 \leq s \end{cases}$$

$$\tilde{\sigma}_{ri}(s) = \begin{cases} 100s^4 & , \quad 0 \leq s < 0.1 \\ s^2 & , \quad 0.1 \leq s \end{cases}, \ i = 1, 2 \ .$$

For $S_i(|x_i|) = |x_i|^{3/2}$, $i = 1, 2$, these systems satisfy (1) with (2). In the case of $\beta = \infty$, the $\dot{x}_1$-equation and $\tilde{\sigma}_2$ become

$$\dot{x}_1 = \frac{8}{3}\left\{-|x_1|^{\frac{1}{2}} + (\tilde{\sigma}_1(|u_1|)+\tilde{\sigma}_{r1}(|r_1|)))^{\frac{1}{4}}\right\}x_1, \ x_1 \in \mathbb{R}^2$$

$$\tilde{\sigma}_2(s) = \begin{cases} 10^4\gamma^2 s^8 & , \quad 0 \leq s < 0.1 \\ \gamma^2 s^4 & , \quad 0.1 \leq s \end{cases}$$

and therefore fulfill (1). Trajectories of the interconnected system Σ given by (10) and (11) with the initial condition $x_1(0) = [-2, 2]^T$, $x_2(0) = [3, -1]^T$ are plotted in Fig.2 for $\gamma = 0.9$. $\gamma = 1$ and $\gamma = 1.1$ in each case of $\beta = 1$ and $\beta = \infty$. The six plots suggest that the equilibrium $[x_1^T, x_2^T]^T = 0$ of the interconnection Σ with $r_1(t) \equiv 0$ and $r_2(t) \equiv 0$ is GAS , i.e., Σ is 0-GAS, if and only if $\gamma < 1$, which will be confirmed theoretically by the main results of this paper. Note that the 0-GAS is necessary for iISS and ISS. A single formula employed in this paper gives a pair $\{\Sigma_1, \Sigma_2\}$ achieving the instability of the interconnection whenever $\gamma \geq 1$ holds for each $\beta \in (0, \infty]$, which again deals with iISS and ISS in a unified manner. The previously existing results can only treat ISS and iISS separately with mutually different Lyapunov functions [10], [11], [14].

The condition (3) with c_1, $c_2 > 1$ always implies

$$\alpha_1^{-1} \circ \sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(s) < s, \quad \forall s \in (0, \infty) \ , \tag{12}$$

while the condition (12) does not guarantee the existence of c_1, $c_2 > 1$ achieving (3). To illustrate (12) which is milder than the existence of c_1, $c_2 > 1$ attaining (3), now consider (1) with the following choice of supply rate functions:

$$\alpha_1(s) = \frac{\beta s^2}{s^2+\beta}, \quad \sigma_1(s) = \frac{\beta s^2}{s^2+2\beta}, \quad \sigma_{r1}(s) = 0 \tag{13}$$

$$\alpha_2(s) = \left(\frac{\beta s^2}{s^2+2\beta}\right)^2, \ \sigma_2(s) = \left(\frac{\gamma\beta s^2}{\gamma s^2+\beta}\right)^2, \ \sigma_{r2}(s) = 0. \tag{14}$$

For each $\beta \in (0, \infty]$, these functions fulfill (12) if and only if $\gamma < 1$. However, the inequality (12) is never achieved in a uniform manner. Indeed, the gap between the functions at both sides of (12) approaches zero as s tends to infinity for all $\gamma < 1$. The condition (3) can be regarded as a special form of

$$\alpha_1^{-1} \circ (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{15}$$

when $(\mathbf{Id} + \omega_i)(s) = c_i s$, $i = 1, 2$. The generalized condition (12) allows ω_i(s)'s to be nonlinear. In contrast, the earlier study in [10], [11] imposes the uniform contraction on the loop gain, i,e,. $(\mathbf{Id} + \omega_i)(s) = c_i s$. The inequality (15) is satisfied for (13) and (14) with nonlinear ω_i(s)'s as follows:

$$\begin{aligned} &\omega_i(s) = \tau_i(s) - s, \quad \omega_i \in \mathcal{P}_0, \quad i = 1, 2 \\ &\tau_1(s) = \alpha_1 \circ \chi_1^{-1}(s) > s, \quad \forall s \in (0, \beta) \\ &\tau_2(s) = [\chi_2 \circ \sigma_2^{-1}(s)]^2 > s, \ \forall s \in (0, \beta^2) \\ &\tau_1(s) = s, \ \forall s \geq \beta, \quad \tau_2(s) = s, \ \forall s \geq \beta^2 \\ &\chi_i(s) = \alpha_1(s)\sqrt{\frac{s^2+\beta}{s^2 + \dfrac{\beta}{1+\zeta_i(\gamma-1)}}}, \quad \begin{array}{l} \zeta_1 = 0.99 \\ \zeta_2 = 1 \quad . \end{array} \end{aligned} \tag{16}$$

According to a result of this paper, a smooth Lyapunov function verifying 0-GAS can be obtained in the form of (4) for all $\beta \in (0, \infty]$ if $\gamma < 1$. The integrands $\lambda_1(s)$ and $\lambda_2(s)$ similar to (5)-(9) are computed by replacing (2) and c_i with (13)-(14) and τ_i. Even for the supply rates (13)-(14) which never result in a loop gain of uniform contraction, we can explicitly obtain a pair $\{\Sigma_1, \Sigma_2\}$ whose interconnection is not 0-GAS for each $\beta \in (0, \infty]$ whenever $\gamma \geq 1$.

III. System Description

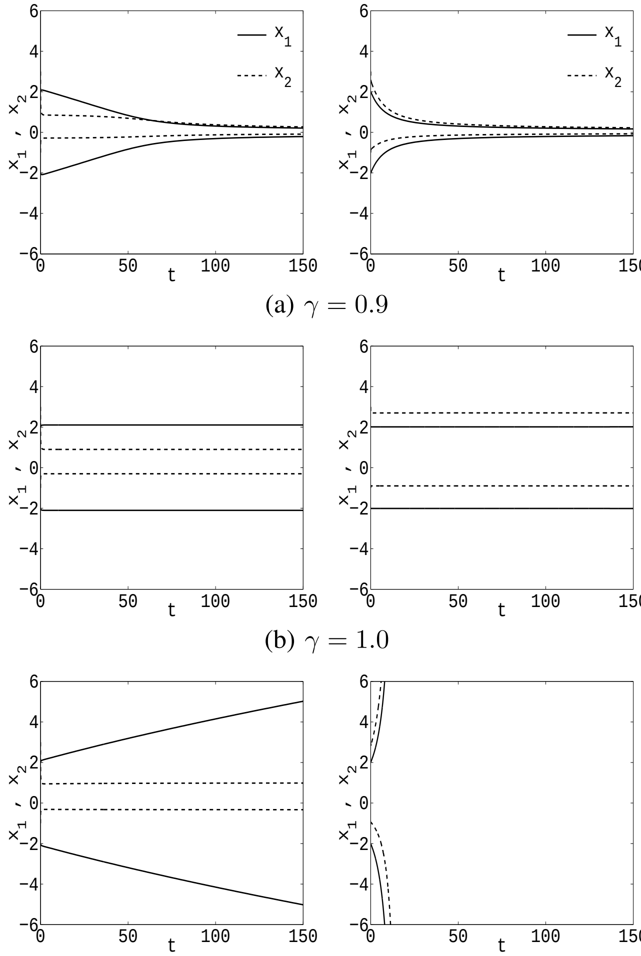
Consider the interconnected system Σ shown in Fig.1. The subsystems Σ_1 and Σ_2 are connected with each other through $u_1 = x_2$ and $u_2 = x_1$. The state vector of Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The signals r_1 and r_2 are packed into $r = [r_1^T, r_2^T]^T \in \mathbb{R}^k$. The following sets of Σ_i's are considered in this paper.

Definition 1: Given $\alpha_i \in \mathcal{P}$, $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{P}_0$ for $i = 1, 2$, let $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i = 1, 2$ denote the pair of sets containing systems Σ_i in the form of

$$\dot{x}_i = f_i(t, x_i, u_i, r_i), \ x_i \in \mathbb{R}^{n_i}, \ u_i \in \mathbb{R}^{m_i}, \ r_i \in \mathbb{R}^{k_i} \tag{17}$$

$$f_i(t, 0, 0, 0) = 0, \qquad t \in \mathbb{R}_+ \tag{18}$$

$$f_i \text{ is locally Lipschitz in } (x_i, u_i, r_i) \text{ and piecewise continuous in } t \tag{19}$$



(a) $\gamma = 0.9$

(b) $\gamma = 1.0$

(c) $\gamma = 1.1$

$\beta = 1$ $\beta = \infty$

Fig. 2. Response of state variables of the system (10) and (11).

for which there exist $\mathbf{C}^1$ functions V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t, x_i) \le \bar{\alpha}_i(|x_i|) \tag{20}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1 \le -\alpha_1(V_1(t,x_1)) + \sigma_1(V_2(t,x_2)) + \sigma_{r1}(|r_1|) \tag{21}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2 \le -\alpha_2(V_2(t,x_2)) + \sigma_2(V_1(t,x_1)) + \sigma_{r2}(|r_2|) \tag{22}$$

hold for all $x_i \in \mathbb{R}^{n_i}$, $r_i \in \mathbb{R}^{k_i}$ and $t \in \mathbb{R}_+$, $i = 1, 2$.

The integers m_i's are supposed to satisfy $m_1 = n_2$ and $m_2 = n_1$ so that the interconnection of Σ_1 and Σ_2 makes sense. The Lipschitzness imposed on f_i guarantees the existence of a unique maximal solution of Σ for locally essentially bounded $r_i(t)$. If the exogenous signal r_i is absent, the set of systems is denoted by $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i)$.

The inequalities (21) and (22) are often referred to as "dissipation inequalities", and their right hand sides are called supply rates. The individual system Σ_i fulfilling the above definition is said to be integral input-to-state stable (iISS) [23]. The function V_i is called a $\mathbf{C}^1$ iISS Lyapunov function [2]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is said to be input-to-state stable (ISS) [22], and the function V_i is a $\mathbf{C}^1$ ISS Lyapunov function [24]. By definition, an ISS system is always iISS. The converse does not hold. The original notion of iISS and ISS is given in terms of trajectories and, in the context of time-invariant systems, is equivalent to the existence of $\mathbf{C}^1$ iISS and ISS Lyapunov functions, respectively [2], [24]. As we see on the right hand side of (21) and (22), the iISS and ISS properties we consider in this paper are uniform in time t. Notions of non-uniform ISS are available in [18] and references therein. Definitions of ISS for discrete-time systems are also available in the literature, e.g., [16], [17].

Definition 2: Given $\alpha_i \in \mathcal{P}$, $\sigma_i \in \mathcal{K}$, $\sigma_{ri} \in \mathcal{P}_0$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ for $i = 1, 2$, let $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ denote the set of systems Σ_i of the form (17), (18) and (19) which admit the existence of a $\mathbf{C}^1$ function V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying (20) and

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i \le -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{23}$$

for all $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $r_i \in \mathbb{R}^{k_i}$ and $t \in \mathbb{R}_+$, $i = 1, 2$.

Definition 3: Let $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ denote the set of Σ_i for which there exist $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ such that $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ holds.

We write $\mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ and $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \underline{\alpha}_i, \bar{\alpha}_i)$ when we consider $r_i(t) \equiv 0$. Definitions 2 and 3 involve $|\cdot|$ to measure the magnitude of feedback signals in the dissipation inequalities. As we will see in the sequel, for the set $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ whose dissipation inequalities do not involve the Euclidean norm of feedback signals, stability criteria become simpler than those for $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$. The set $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ in Definition 3 naturally generalizes the notion of prescribed $\mathcal{L}^p$-gain systems. By comparison, the set $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ in Definition 2 includes the explicit information $\underline{\alpha}_i$, $\bar{\alpha}_i$ on the discrepancy between $|\cdot|$ and $V_i(\cdot)$, which is essential to the analysis of 0-GAS of the interconnection.

IV. Main Results

The following theorem provides a necessary and sufficient condition for the uniform 0-GAS of a set of interconnected iISS systems as shown in Fig. 1. By uniform 0-GAS, we mean that the trivial solution of the interconnected system Σ without external inputs r_1 and r_2 is uniformly GAS.

Theorem 1: Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ are $\mathbf{C}^1$ and satisfy

$$\alpha_i \in \mathcal{O}(>1), \ \sigma_i \in \mathcal{O}(>0), \quad i = 1, 2 \tag{24}$$

$$\alpha_i \in \mathcal{K}, \quad i = 1, 2. \tag{25}$$

Suppose that there exists some integer $j \in \{1, 2\}$ such that α_1, α_2, σ_1 and σ_2 satisfy

$$\lim_{s\to\infty} \alpha_{3-j}(s) \ge \lim_{s\to\infty} \sigma_{3-j}(s) \tag{26}$$

and one of the following conditions

(G1) $\lim_{s\to\infty} \alpha_{3-j}(s) = \lim_{s\to\infty} \sigma_{3-j}(s)$

(G2) $\displaystyle\lim_{s\to\infty}\frac{\sigma_j\circ\alpha_{3-j}^{-1}\circ\sigma_{3-j}(s)}{\alpha_j(s)}\neq 1$.

Then, the interconnected system Σ is uniformly 0-GAS for all pairs $\Sigma_i\in\mathcal{S}_{Vi}(n_i,\alpha_i,\sigma_i)$, $i=1,2$ if and only if

$$\alpha_j^{-1}\circ\sigma_j\circ\alpha_{3-j}^{-1}\circ\sigma_{3-j}(s)<s,\quad \forall s\in(0,\infty) \tag{27}$$

holds for the above j. Furthermore, a Lyapunov function of Σ characterizing the uniform 0-GAS is given as

$$V_{cl}(t,x)=\int_0^{V_1(t,x_1)}\lambda_1(s)ds+\int_0^{V_2(t,x_2)}\lambda_2(s)ds \tag{28}$$

for some non-decreasing continuous functions $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$\lambda_1(s)>0,\ \lambda_2(s)>0,\quad s\in(0,\infty)\ . \tag{29}$$

It is emphasized that j in (27) is the same as in any of *(G1)*-*(G2)*. The properties (25) and (26) are assumed beforehand only for simplicity of expressions. Their necessity will be proven in Theorem 4 and Theorem 5 of Section V. It is stressed that *(G1)*-*(G2)* are not simultaneous constraints. Only one of them is required. Let the inequality (27) be referred to as a small-gain condition. It is mentioned here that the uniform 0-GAS in Theorem 1 is derived from

$$\exists\alpha_{cl}\in\mathcal{P}\ \text{ s.t. }\ \dot V_{cl}(t,x)\le-\alpha_{cl}(|x|),\quad \forall x\in\mathbb{R}^n \tag{30}$$

satisfied along the trajectories of the interconnected system Σ with $r_i(t)\equiv 0$, $i=1,2$.

One can obtain iISS of a set of interconnected systems if amplification factors ω_i, $i=1,2$, are introduced to the small-gain condition. A stronger property, ISS, is a special case.

Theorem 2: Assume that functions $\alpha_i,\sigma_i,\sigma_{ri},\underline{\alpha}_i,\bar\alpha_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$ satisfy (25). Suppose that there exists some integer $j\in\{1,2\}$ such that one of the following conditions

(H1) $\displaystyle\lim_{s\to\infty}\alpha_1(s)=\infty\quad\wedge\quad\lim_{s\to\infty}\alpha_2(s)=\infty$

(H2) $\displaystyle\lim_{s\to\infty}\alpha_{3-j}(s)=\infty\quad\wedge\quad\lim_{s\to\infty}\sigma_{3-j}(s)<\infty$

(H3) $\displaystyle\lim_{s\to\infty}\sigma_1(s)<\infty\quad\wedge\quad\lim_{s\to\infty}\sigma_2(s)<\infty$

is satisfied. Then, the interconnected system Σ is iISS with respect to input r and state x for all pairs $\mathcal{S}_i(n_i,\alpha_i,\sigma_i,\sigma_{ri},\underline{\alpha}_i,\bar\alpha_i)$ with any positive integer n_i, $i=1,2$ if there exist $\omega_i\in\mathcal{K}_\infty$, $i=1,2$ such that

$$\begin{aligned}&\underline{\alpha}_j^{-1}\circ\bar\alpha_j\circ\alpha_j^{-1}\circ(\mathbf{Id}+\omega_j)\circ\sigma_j\\ &\quad\circ\underline{\alpha}_{3-j}^{-1}\circ\bar\alpha_{3-j}\circ\alpha_{3-j}^{-1}\circ(\mathbf{Id}+\omega_{3-j})\circ\sigma_{3-j}(s)\le s\ ,\\ &\qquad\qquad\qquad\qquad\forall s\in\mathbb{R}_+\end{aligned} \tag{31}$$

holds for the above j. Furthermore, an iISS Lyapunov function of Σ is given as in (28) for some non-decreasing continuous functions $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ satisfying (29). In the case of *(H1)*, the function V_{cl} is also an ISS Lyapunov function.

Note that the inverses of α_j and α_{3-j} in (27) and (31) are not necessarily well defined over $\mathbb{R}_+$. Instead, the fulfillment of (27) and (31) only requires the whole composite function on the left hand side of the inequality to be finite for finite s. Thus, $\lim_{s\to\infty}\alpha_j(s)\ge\lim_{s\to\infty}\sigma_j(s)$ is not necessary. The statement about a Lyapunov function in Theorem 2 claims that

$$\begin{aligned}&\exists\alpha_{cl}\in\mathcal{P},\ \sigma_{cl}\in\mathcal{P}_0\ \text{ s.t.}\\ &\quad\dot V_{cl}(t,x)\le-\alpha_{cl}(|x|)+\sigma_{cl}(|r|),\ \forall x\in\mathbb{R}^n,r\in\mathbb{R}^k\end{aligned} \tag{32}$$

is satisfied along the trajectories of Σ. Since the above theorem only addresses the sufficiency of a small-gain condition for the stability, neither (24) nor the smoothness of α_i and σ_i is required. It is stressed that j in (31) is the same as in *(H2)*. It can be verified that

$$\textit{(G1)}\vee\textit{(G2)}\Leftarrow\textit{(H1)}\vee\textit{(H2)}\vee\textit{(H3)}\Leftarrow\textit{(H1)}$$

holds under the assumption that there exist $\omega_1,\omega_2\in\mathcal{K}_\infty$ satisfying (31).

Theorem 3: Let n_i be a positive integer for each $i=1,2$. Assume that functions $\alpha_i,\sigma_i,\sigma_{ri}:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$ are $\mathbf{C}^1$ and satisfy (24), (25), *(H1)* and

$$\sigma_{ri}\in\mathcal{K}_\infty,\quad i=1,2\ . \tag{33}$$

Then, the interconnected system Σ is ISS with respect to input r and state x for all pairs $\mathcal{S}_{Vi}(n_i,\alpha_i,\sigma_i,\sigma_{ri})$, $i=1,2$ if and only if there exist $\omega_i\in\mathcal{K}_\infty$, $i=1,2$ such that

$$\begin{aligned}&\alpha_1^{-1}\circ(\mathbf{Id}+\omega_1)\circ\sigma_1\circ\alpha_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(s)\le s\ ,\\ &\qquad\qquad\qquad\forall s\in\mathbb{R}_+\end{aligned} \tag{34}$$

holds. Furthermore, an ISS Lyapunov function of Σ is given as in (28) for some non-decreasing continuous functions $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ satisfying (29).

Theorem 3 indicates that there exists $\alpha_{cl}\in\mathcal{K}_\infty$ achieving (32). In contrast to Theorem 2 stated with $\sigma_{ri}\in\mathcal{P}_0$, Theorem 3 considers (33) which is narrower than $\mathcal{P}_0$. The assumption (33) is only for obtaining the "only if" part of Theorem 3. If the exogenous signals affect systems through sufficiently small $\sigma_{ri}\notin\mathcal{K}_\infty$, the condition (34) is not always required, while (27) is necessary. For sufficiently small σ_{r1} and σ_{r2}, none of *(H1)*, *(H2)* and *(H3)* is necessary (See Section VI-A).

It is stressed that (31) with $j=1$ is not equivalent to (31) with $j=2$ in general. The same remark applies to (27). The $j=1$ case in (31) implies the $j=2$ case if

$$\lim_{s\to\infty}\alpha_1(s)=\infty\ \vee\ \lim_{s\to\infty}\alpha_1(s)>\lim_{s\to\infty}\sigma_1(s)\ . \tag{35}$$

Thus, the condition (31) is symmetric in terms of $j=1$ and $j=2$ when Σ_1 and Σ_2 are individually ISS with respect to the interacting inputs. When iISS subsystems are involved, we need to select $j\in\{1,2\}$ or interchange Σ_1 and Σ_2 so that (31) or (27) can be fulfilled. Theorem 4 in Section V explains why the condition should be asymmetric.

Combining the results in Sections V and VI proves the theorems in this section.

Remark 1: In the case of *(H1)*, the inequalities (31) and (34) are in accordance with the nonlinear small-gain condition proposed by [15] when the gains of individual subsystems are computed from differential dissipation inequalities, i.e., α_i and σ_i. The nonlinear small-gain condition was originally a sufficient condition for stability of interconnected ISS systems. This paper unifies the treatment of iISS and ISS systems. Note again that (24) and the smoothness of α_i and σ_i are needed only for the necessity part. The slight difference between (31) and (34) arises from the difference of $\mathcal{S}_i$ and $\mathcal{S}_{Vi}$ as mentioned at the end of Section III.

Remark 2: In contrast to an earlier result of the first author [11], theorems in this paper do not require the small-gain conditions to be uniform contraction over $\mathbb{R}_+$. In other words, linearity is not imposed on the amplification factors $\omega_i(s)$, $i=1,2$, in the small-gain conditions of (31) and (34). This comment also applies to (27) in view of the existence of ω_i. This paper demonstrates that the uniformity assumption on the small-gain conditions, i.e., restricting $\omega_i(s)$ to $(c_i-1)s$ for a constant $c_i>1$, in [11] can be removed in the explicit construction of smooth Lyapunov functions.

Remark 3: The small-gain condition (31) is simpler and less restrictive than the small-gain condition proposed in [11]. Indeed, the small-gain-like condition in [11] for iISS systems is given as the existence of $k>0$ and c_1, $c_2>1$ satisfying

$$\max_{w\in[0,s]}\frac{[c_j\sigma_j\circ\underline{\alpha}_{3-j}^{-1}\circ\bar{\alpha}_{3-j}\circ\underline{\alpha}_{3-j}^{-1}\circ c_{3-j}\sigma_{3-j}(w)]^k}{\sigma_{3-j}(w)} \leq\frac{[\alpha_j\circ\bar{\alpha}_j^{-1}\circ\underline{\alpha}_j(s)]^k}{\sigma_{3-j}(s)},\quad \forall s\in\mathbb{R}_+ \quad (36)$$

under the assumption of

$$\lim_{s\to\infty}\alpha_{3-j}(s)=\infty\vee \{\lim_{s\to\infty}\alpha_{3-j}(s)>\lim_{s\to\infty}\sigma_{3-j}(s)\wedge\lim_{s\to\infty}\sigma_j(s)<\infty\}. \quad (37)$$

In contrast, this paper proves that the single small-gain condition (31) applies to ISS systems and iISS systems equally. The existence of $k>0$ and c_1, $c_2>1$ fulfilling (36) implies (31) with $s+\omega_i(s)=c_is$. On the other hand, the existence of $k>0$ and c_1, $c_2>1$ fulfilling (36) is not guaranteed even if (31) is satisfied with linear ω_i's. For example, the following functions:

$$\underline{\alpha}_i=\bar{\alpha}_i,\ i=1,2$$
$$\alpha_1(s)=\frac{3s}{2(s+1)},\quad \sigma_1(s)=s$$
$$\alpha_2(s)=\begin{cases}0 & ,\ s=0\\ s^{\frac{1}{s}} & ,\ 0<s\leq 1\\ s & ,\ 1<s\end{cases}$$
$$\sigma_2(s)=\begin{cases}0 & ,\ s=0\\ \dfrac{1}{2}\left(\dfrac{s}{s+1}\right)^{\frac{s+1}{s}} & ,\ s>0\end{cases}$$

satisfy the condition (31) as

$$\frac{3}{2}\sigma_1\circ\alpha_2^{-1}\circ\frac{3}{2}\sigma_2(s)\leq\alpha_1(s),\ \forall s\in\mathbb{R}_+ .$$

It can be, however, verified that

$$\lim_{s\to 0^+}\frac{[c_1\sigma_1\circ\alpha_2^{-1}\circ c_2\sigma_2(s)]^k}{\sigma_2(s)} =\lim_{s\to 0^+}c_1^kc_2s^{\left(k-\frac{1}{s}\right)}=\infty,\ \forall k>0$$

holds. Since the left hand side of (36) is not finite for all k, there are no c_1, $c_2>1$ satisfying (36). Hence, the previous result (36) is more restrictive than (31) even when (31) is used with linear ω's. Moreover, under (36), the situation (37) is covered completely by *(H1)* $\vee$ *(H2)* $\vee$ *(H3)*.

Remark 4: All the theorems given above use a single pair of continuous functions $\{\lambda_1(s),\lambda_2(s)\}$ to generate Lyapunov functions V_{cl}, which will be shown explicitly in Section VI. Hence, a common form of Lyapunov functions can be used for both iISS and ISS systems. This unification of Lyapunov functions for iISS and ISS systems is a novelty of this paper. The previous study [11] employs a Lyapunov function for iISS systems which is different from that for ISS systems.

Remark 5: The constraint *(G1)* $\vee$ *(G2)* is not a technical assumption. Consider

$$\alpha_1(s)=\frac{s}{1+s},\ \sigma_1(s)=s,\ \sigma_2(s)=\alpha_2\left(\frac{s^2}{1+s+s^2}\right) \quad (38)$$

$$\alpha_2(s)=\begin{cases}2^{3-p}(p-2)s^3 \\ \quad +2^{2-p}(3-p)s^2 & ,\ s\in[0,1/2)\\ s^p & ,\ s\in[1/2,\infty)\ .\end{cases} \quad (39)$$

These class $\mathcal{K}$ functions are $\mathbf{C}^1$, and satisfy $\alpha_i\in\mathcal{O}(>1)$ and $\sigma_i\in\mathcal{O}(>0)$ for $0<p\leq 3$. Define

$$\dot{x}_1=-\alpha_1(x_1)+\sigma_1(x_2) \quad (40)$$
$$\dot{x}_2=-\alpha_2(x_2)+\sigma_2(x_1) \quad (41)$$
$$V_1=x_1,\quad V_2=x_2 \quad (42)$$

for the non-negative initial conditions $(x_1(0),x_2(0))\in\mathbb{R}_+^2$. Due to $\alpha_i,\sigma_i\in\mathcal{K}$, the set $\mathbb{R}_+^2$ is positively invariant. Therefore, $V_1=x_1$ and $V_2=x_2$ are positive definite and $\mathbf{C}^1$ on $\mathbb{R}_+^2$ where $x(t)$ evolves, and satisfy (21) and (22). The interconnected system (40)-(41) belongs to the class of positive systems which are popular in biological and chemical processes. This system (40)-(41) has unbounded trajectories in the case of $0<p<1$ [1]. However, α_i, σ_i, $i=1,2$, in (38) and (39) satisfy (27) with $j=1$. Indeed, the functions are one of the cases where $\neg$*(G1)* $\wedge$ $\neg$*(G2)* is fulfilled. Thus, the assumption cannot be eliminated from Theorem 1.

Remark 6: In Theorems 1 and 3, the pair $\mathcal{S}_i(n_i,\alpha_i,\sigma_i,\sigma_{ri})$, $i=1,2$ can replace $\mathcal{S}_{Vi}(n_i,\alpha_i,\sigma_i,\sigma_{ri})$, $i=1,2$ if $\underline{\alpha}_i=\bar{\alpha}_i$ holds for $i=1,2$.

Remark 7: The "only if" part of Theorem 1 does not need the assumption *(G1)* $\vee$ *(G2)*. In other words, there always exists a pair of Σ_i, $i=1,2$ such that their interconnection is not 0-GAS when (27) is violated.

V. Necessity

The issue of the necessity of stability criteria is important from the perspective of estimating stability margins for uncertain systems as well as the tightness of the stability criteria.

A. Destabilizing Perturbation

The following is a new technique to construct destabilizing perturbations, which is the key to the proof of the necessity in Theorems 1 and 3.

Lemma 1: Suppose that $\mathbf{C}^1$ functions $\alpha\in\mathcal{P}$, $\sigma\in\mathcal{K}$, real numbers $\delta\geq 0$, $\bar{\epsilon}>0$ and integers $n>0$, $m>0$ are given. Assume that α and σ belong to $\mathcal{O}(>1)$ and $\mathcal{O}(>0)$, respectively. Then, there exist a locally Lipschitz function f: $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, a $\mathbf{C}^1$ function V: $\mathbb{R}^n\to\mathbb{R}$, class $\mathcal{K}_\infty$

functions $\underline{\alpha}$, $\bar{\alpha}$ and a real number $\epsilon \in [0, \bar{\epsilon}]$ such that

$$f(0,0) = 0 \tag{43}$$

$$\underline{\alpha}(|x|) = V(x) = \bar{\alpha}(|x|) \tag{44}$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(|x|) + \sigma(|u|),\ \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{45}$$

$$\left. \begin{array}{r} (1+\delta)\alpha(|x|) < \sigma(|u|) \\ \epsilon \leq |u| \end{array} \right\} \Rightarrow \frac{\partial V}{\partial x} f(x,u) > \delta\alpha(|x|)\ . \tag{46}$$

The proof of this lemma given in Appendix A is constructive. The functions f_i, $i = 1, 2$, in Section II are constructed by using this technique, so that the interconnected system Σ composed of Σ_1 and Σ_2 in (10) and (11) becomes unstable in the case of $\gamma \geq 1$.

Remark 8: When $(1/N) + (1/J) < 1$, $\alpha \in \mathcal{O}(N)$ and $\sigma \in \mathcal{O}(J)$ are satisfied, the claim in Lemma 1 still holds for $\bar{\epsilon} = 0$.

Remark 9: The function $f(x, u)$ constructed in the proof of Lemma 1 satisfies

$$f_i(x,u)|_{x_i=0} = 0, \quad i = 1, 2, \ldots, n\ ,$$

where $f = [f_1, f_2, ..., f_n]^T$. This implies that each i-th scalar component of the solution vector $x(t) \in \mathbb{R}^n$ of the differential equation $\dot{x} = f(x, u)$ never changes signs, namely, for each $i = 1, 2, \ldots, n$,

$$x_i(0) \geq 0 \ \Rightarrow\ x_i(t) \geq 0,\ \forall t \in \mathbb{R}_+$$

holds. For such a positive system defined for initial conditions in the non-negative orthant, the $\mathbf{C}^1$ function $V(x)$ needs to be defined on only $\mathbb{R}^n_+$. Since $V(x) = |x|$ becomes eligible, Lemma 1 allows $\alpha \in \mathcal{O}(1)$ when one's attention is restricted to positive systems. Finally, it can be verified that all the results in this paper hold even for the interconnection of subsystems evolving on $\mathbb{R}^{n_i}_+$.

Remark 10: For linear uncertain systems, the techniques for proving the necessity of small-gain conditions are widely known, and consist of constructing a destabilizing perturbation(or a particular uncertain system) whenever a linear small-gain condition is violated. Unfortunately, the popular linear approach does not allow us to select the order of the destabilizing system [6], [28], and it requires the notion of output. The minimal order is in general the sum of dimension of input and output vectors, and the order is larger than the output dimension of the destabilizing perturbation. On the other hand, the technique proposed in Lemma 1 not only enables us to deal with nonlinear gains, but also allows us to choose the order of the destabilizing system arbitrarily. For instance, a destabilizing perturbation can be given by a system of dimension one making use of nonlinearities. Furthermore, providing a Lyapunov characterization of the destabilizing system is a notable feature of the proposed technique.

B. Necessary Conditions

Using Lemma 1, we can derive necessary conditions for the stability of the interconnected system Σ shown in Fig.1. The following addresses the existence of an integer $j \in \{1, 2\}$ satisfying (26) which is required in all developments in this paper.

Theorem 4: Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$, and satisfy

$$\alpha_i \in \mathcal{O}(>1), \quad \sigma_i, \sigma_{ri} \in \mathcal{O}(>0), \quad i = 1, 2\ . \tag{47}$$

Then, for the pair

$$\begin{aligned} S_i = \{\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}) : \\ f_i(t, x_i, u_i, r_i) = f_i(0, x_i, u_i, r_i),\ \forall t \in \mathbb{R}_+\}\ , \\ i = 1, 2 \end{aligned} \tag{48}$$

and the pair

$$\begin{aligned} S_i = \{\Sigma_i \in \mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri}) : \\ f_i(t, x_i, u_i, r_i) = f_i(0, x_i, u_i, r_i),\ \forall t \in \mathbb{R}_+\}\ , \\ i = 1, 2 \end{aligned} \tag{49}$$

the following facts hold.

(i) The interconnected system Σ is 0-GAS for all $\Sigma_i \in S_i$, $i = 1, 2$, only if

$$\liminf_{s\to\infty} \alpha_i(s) \geq \lim_{s\to\infty} \sigma_i(s) \tag{50}$$

holds for at least one of $i = 1, 2$.

(ii) The interconnected system Σ is ISS with respect to input r and state x for all $\Sigma_i \in S_i$, $i = 1, 2$, only if

$$\liminf_{s\to\infty} \alpha_i(s) \geq \lim_{s\to\infty} \sigma_i(s) + \sup_{s\in\mathbb{R}_+} \sigma_{ri}(s) \tag{51}$$

holds for at least one of $i = 1, 2$.

The necessary condition (51) and (33) justify either of the two requirements in *(H1)* of Theorem 3. The use of the sets (48) and (49) illustrates that the necessity holds for sets of time-invariant systems which are narrower than $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, respectively. Note that (50) is also necessary for iISS of Σ since iISS implies 0-GAS. The property (51) indicates that Σ_i is ISS with respect to input (u_i, r_i) and state x_i if $\sigma_{ri} \in \mathcal{K}$. The property (50) implies that Σ_i is ISS with respect to input u_i and state x_i. It is worth noting that $\limsup_{s\to\infty} \sigma_{ri}(s) < \infty$ is not necessary for the iISS property of the interconnected system Σ even if $\liminf_{s\to\infty} \alpha_i(s) < \infty$. This fact can be understood naturally. In fact, a system is iISS if and only if it is 0-GAS and zero-output smoothly dissipative [2].

On the basis of this result, we can establish the necessity of the small-gain condition.

Theorem 5: Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$, and satisfy (47). Suppose

$$\begin{aligned} &\liminf_{s\to\infty} \alpha_2(s) = \infty\ \vee \\ &\qquad \left\{ \begin{array}{ll} \liminf\limits_{s\to\infty} \alpha_2(s) \geq \lim\limits_{s\to\infty} \sigma_2(s) & \text{if } 2 \notin \mathbf{D} \\ \liminf\limits_{s\to\infty} \alpha_2(s) > \lim\limits_{s\to\infty} \sigma_2(s) & \text{if } 2 \in \mathbf{D} \end{array} \right. \end{aligned} \tag{52}$$

holds, where $\mathbf{D} := \{i \in \{1, 2\} : \sigma_{ri} \in \mathcal{K}_\infty\}$. Then, the following facts hold for the pairs S_1, S_2 defined in (48) and (49).

(i) The interconnected system Σ is 0-GAS for all $\Sigma_i \in S_i$, $i = 1, 2$ only if there exist $\tilde{\alpha}_i \in \mathcal{K}$, $i = 1, 2$, such that

$$\tilde{\alpha}_1^{-1} \circ \sigma_1 \circ \tilde{\alpha}_2^{-1} \circ \sigma_2(s) < s \quad \forall s \in (0, \infty) \tag{53}$$

$$\tilde{\alpha}_i(s) \leq \alpha_i(s), \quad \forall s \in \mathbb{R}_+ \ . \tag{54}$$

(ii) The interconnected system Σ is ISS with respect to input r and state x for all $\Sigma_i \in S_i$, $i = 1, 2$ only if there exist

$$\omega_i \begin{cases} \in \mathcal{K}_\infty & \text{if } i \in \mathbf{D} \\ = 0 & \text{if } i \notin \mathbf{D} \end{cases} \tag{55}$$

and $\tilde{\alpha}_i \in \mathcal{K}$ for $i = 1, 2$ such that

$$\tilde{\alpha}_1^{-1} \circ (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \tilde{\alpha}_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \leq s \ , \quad \forall s \in \mathbb{R}_+ \tag{56}$$

and (54) are satisfied.

Note that we can take $\tilde{\alpha}_i = \alpha_i$ if α_i is of class $\mathcal{K}$. Theorem 5 suggests that we can assume $\alpha_i \in \mathcal{K}$, $i = 1, 2$, without loss of generality in the stability analysis. The next lemma indicates that the assumption of $\alpha_i \in \mathcal{O}(> 1)$ and $\sigma_i \in \mathcal{O}(> 0)$ is reasonable.

Lemma 2: For $n_i > 0$, the following holds.

(i) If $\partial V_i/\partial x_i$ and $\partial V_i/\partial t$ are Hölder continuous of some order $a > 0$ and $b > 1$, respectively, in x_i at $x_i = 0$, then $\mathcal{S}_i(n_i, \alpha_i, \sigma_i) \neq \emptyset$ implies $\alpha_i \in \mathcal{O}(> 1)$.

(ii) For each $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$, there exists $\hat{\sigma}_i \in \mathcal{K}$ such that $\hat{\sigma}_i \in \mathcal{O}(> 0)$ and $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \hat{\sigma}_i) \subseteq \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ holds.

C. Discussions

The ISS property of a subsystem Σ_i with respect to its feedback input u_i and its state x_i is not necessary for 0-GAS if the specific differential equations describing Σ_1 and Σ_2 are given instead of merely dissipation inequalities.

Fact 1: There exists a pair of iISS subsystems Σ_1 and Σ_2 fulfilling the following simultaneously.

(i) Each Σ_i, $i = 1, 2$, is not ISS.

(ii) The interconnected system Σ is uniformly 0-GAS.

This fact can be confirmed by

$$\Sigma_1 \ : \ \dot{x}_1 = -\mathrm{sat}(x_1) + x_2 \tag{57}$$

$$\Sigma_2 \ : \ \dot{x}_2 = -\mathrm{sat}(x_2) - x_1 \ . \tag{58}$$

The 0-GAS of this interconnection follows from $V_{cl} = x_1^2 + x_2^2$. Although $V_i = \log(1 + x_i^2)$ proves that each Σ_i is iISS with respect to state x_i and input x_{3-i}, the two subsystems are not ISS. The stability property *(ii)* of Σ in Fact 1 can be even strengthened to ISS by adding $+x_1 r_1/(1 + x_1^2)$ and $+x_2 r_2/(1 + x_2^2)$ to (57) and (58), respectively. In contrast, Theorem 4 has demonstrated that the ISS property is necessary in the situation where the information of only dissipation inequalities is available.

Angeli and Astolfi [1] have pointed out that asymptotic stability of a feedback system is not always detected by means of gain conditions alone. Indeed, the small-gain constraint (53) is not necessary if both subsystems are specified directly by differential equations of state variables. Theorem 5 claims the necessity only when the two systems are allowed to be uncertain as long as each subsystem satisfies a given dissipation inequality. In fact, the aforementioned example illustrating Fact 1 violates (53) clearly since two subsystems are only iISS.

An interesting question is whether the small-gain condition is necessary when one of the two subsystems is given and fixed by a differential equation. Unfortunately, for nonlinear systems, the answer is in general negative. The necessity property of the small-gain condition is very delicate due to the diversity of system nonlinearities and supply rates or gain functions to be chosen. The situation is illustrated by the following fact.

Fact 2: There exist a subsystem Σ_2, functions $\alpha_1 \in \mathcal{P}$ and $\alpha_2, \sigma_1, \sigma_2 \in \mathcal{K}$ such that the following is fulfilled simultaneously.

(i) $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_2)$ holds.

(ii) $\Sigma_2 \notin \mathcal{S}_2(n_2, \alpha_2, \mu\sigma_2)$ holds for all $\mu < 1$.

(iii) The interconnected system Σ is is uniformly 0-GAS for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1)$ for all integer $n_1 > 0$.

(iv) There exists $s \in (0, \infty)$ for which it holds that

$$\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(s) \geq \alpha_1(s) \ . \tag{59}$$

This fact suggests that the small-gain condition is not necessarily satisfied when one does not want stability for "all" elements in prescribed sets of systems. One example explaining this fact is

$$\alpha_1(s) = \frac{s^2}{s^2 + 1}, \quad \sigma_1(s) = \frac{s^2}{2} \tag{60}$$

$$\alpha_2(s) = s^2, \quad \sigma_2(s) = s^2 \ . \tag{61}$$

Consider a particular model of Σ_2 given by

$$\Sigma_2 \ : \ \dot{x}_2 = -x_2 + \frac{x_1}{x_1^2 + 1} \tag{62}$$

which satisfies $\Sigma_2 \in \mathcal{S}_2(1, \alpha_2, \sigma_2)$ for $V_2(x_2) = x_2^2$. For the system Σ_1, we only pay attention to the set $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1)$ instead of a particular element. Define $V(x) = V_1(x_1) + cV_2(x_2)$. Then,

$$\dot{V} \leq -(1 - c)\frac{x_1^2}{(x_1^2 + 1)^2} - \left(c - \frac{1}{2}\right) x_2^2$$

is obtained. This inequality with $c = 3/4$ proves the uniform 0-GAS of the interconnected system Σ. The small-gain condition (53) is, however, violated for (60) and (61) since $\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(s) = s^2/2$ implies $\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(s) \geq \alpha_1(s)$ for $s \in [1, \infty)$. The stability is established without the small-gain condition since Σ_2 is fixed at a particular element in the set. The system (62) is on the boundary of the set $\mathcal{S}_2(1, \alpha_2, \sigma_2)$, where the boundary is defined by the product of *(i)* and *(ii)*. The property *(ii)* can be verified for (62) via Jacobian linearization. However, the time-derivative of V_2 is bounded conservatively from above in shape by $\sigma_2(s)$ toward $s \to \infty$. In fact, the system (62) is on the boundary only in the sense of the maximum magnitude. We can strengthen *(iii)* of Fact 2 to iISS by adding $+x_2 r_2/(x_2^2 + 1)$ to (62).

The situation becomes different for supply rates fitting systems tightly such as linear systems with quadratic supply rates. Consider a linearized version of the above example, let (62) be replaced with

$$\Sigma_2 \quad : \quad \dot{x}_2 = -x_2 + x_1 \tag{63}$$

which satisfies *(i)* and *(ii)* of Fact 2 with respect to (61) and $n_2 = 1$. Define a set $\mathcal{S}_1(1, \alpha_1, \sigma_1)$ with

$$\alpha_1(s) = s^2,\ \sigma_1(s) = \gamma s^2 \tag{64}$$

for Σ_1 The interconnection of Σ_1 and Σ_2 is not 0-GAS if Σ_1 is $\dot{x}_1 = -x_1 + x_2$ which satisfies $\Sigma_1 \in \mathcal{S}_1(1, \alpha_1, \sigma_1)$ for $\gamma = 1$. Thus, the interconnected system defined with (63) is uniformly 0-GAS for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1)$ with an arbitrary integer $n_1 > 0$ if and only if $\gamma < 1$. On the other hand, the condition (59) is identical to $\gamma \geq 1$ for (64) and (61). In this way, the properties *(iii)* and *(iv)* cannot be achieved since the linear system Σ_2 in (63) is covered tightly by $\mathcal{S}_2(1, \alpha_2, \sigma_2)$ with the quadratic supply rate (61) which represents a linear gain.

These two examples suggest that the small-gain condition is not necessary for the stability of nonlinear interconnected systems if the supply rate fits the subsystem Σ_2 only in the maximum magnitude. A supply rate bounds an element Σ_2 in a very conservative way if the nonlinearity of Σ_2 is far from the shape of $\{\alpha_2,\sigma_2\}$. It is natural that, in order to obtain the necessity of the small-gain condition, we need to estimate Σ_2 with $\{\alpha_2,\sigma_2\}$ tightly in terms of the shape. The following explains this idea.

Proposition 1: Suppose that $\alpha_1 \in \mathcal{P}$ and $\alpha_2, \sigma_1, \sigma_2 \in \mathcal{K}$ are given, and that $\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \sigma_2(s)$, $\alpha_2 \in \mathcal{O}(>1)$ and $\sigma_2 \in \mathcal{O}(>0)$ hold. Assume that Σ_2 is a system satisfying

(i) $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_2)$ holds.

(ii) There exist a $\mathbf{C}^1$ function V_2: $\mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2$, $\bar{\alpha}_2$ and a positive number l_1 such that

$$\underline{\alpha}_2(|x_2|) = V_2(x_2) = \bar{\alpha}_2(|x_2|) \tag{65}$$

$$\left.\begin{array}{l} |x_1| \geq l_1 \\ |x_2| = \alpha_2^{-1} \circ \sigma_2(l_1) \end{array}\right\} \Rightarrow \frac{\partial V_2}{\partial x_2} f_2(t, x_2, x_1) \geq 0 \tag{66}$$

$$\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(l_1) \geq \alpha_1(l_1) \tag{67}$$

Then, for each integer $n_1 > 0$, there exists a system $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1)$ for which the interconnected system Σ is not 0-GAS.

This proposition follows directly from the proof of Theorem 5. The requirement (66) implies that the pair $\{\alpha_2,\ \sigma_2\}$ fit tightly Σ_2 in shape as well as the maximum magnitude.

VI. Sufficiency

In this section, the sufficiency of the stability criteria presented in Section IV is derived. This section gives a pair of $\{\lambda_1, \lambda_2\}$ with which the composite Lyapunov function V_{cl} in (28) fulfills (30) and (32).

A. A common form of Lyapunov function

Consider the set of the quadruplets $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying

$$\alpha_1, \alpha_2, \sigma_1, \sigma_2 \in \mathcal{K}, \tag{68}$$

$$\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \sigma_2(s)\ . \tag{69}$$

Define the following seven situations for $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$:

(M1) $\displaystyle \lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} \leq 1 \ \wedge \ \lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s)$

(M2) $\displaystyle \lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} < 1 \ \wedge \ \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$

(M3) $\displaystyle \lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} = 1 \ \wedge \ \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$

(J1) $\displaystyle \lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty$

(J2) $\displaystyle \lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

(J3) $\displaystyle \lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

(J4) $\exists k \in \{1, 2\}$ s.t. $\displaystyle \{\lim_{s\to\infty} \alpha_k(s) < \infty \ \wedge \ \lim_{s\to\infty} \sigma_{3-k}(s) = \infty\}$

The pair of $\{\lambda_1, \lambda_2\}$ for the Lyapunov function V_{cl} can be constructed from the functions in the small-gain conditions (27), (31) and (34). The following lemma can be verified straightforwardly, which provides the functions to be used in $\{\lambda_1, \lambda_2\}$ directly.

Lemma 3: Assume that

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s) < s\ , \quad \forall s \in (0, \infty) \tag{70}$$

holds for a pair $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ and a quadruplet $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying (68), (69) and *(M1)* $\vee$ *(M2)*. Then, there exist $\hat{\alpha}_1, \hat{\sigma}_1 \in \mathcal{K}$, $\hat{\omega}_1$, $\hat{\omega}_2 \in \mathcal{P}_0$ and $\hat{\tau}_1, \hat{\tau}_2 \in \mathcal{K}_\infty$ such that

$$(\mathbf{Id} + \hat{\omega}_1) \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \hat{\omega}_2) \circ \sigma_2(s) \leq \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \quad \forall s \in \mathbb{R}_+ \tag{71}$$

$$\sigma_1(s) \leq \hat{\sigma}_1(s), \quad \hat{\alpha}_1(s) \leq \alpha_1(s),\ \forall s \in \mathbb{R}_+ \tag{72}$$

$$\lim_{s\to\infty} \hat{\sigma}_1(s) \geq \lim_{s\to\infty} \hat{\alpha}_1(s) \tag{73}$$

$$\lim_{s\to\infty} \alpha_2(s) = \infty \ \Rightarrow \ \hat{\alpha}_1 = \alpha_1 \tag{74}$$

$$\lim_{s\to\infty} \alpha_2(s) < \infty \ \Rightarrow \ \hat{\sigma}_1 = \sigma_1 \tag{75}$$

$$\lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) \ \Rightarrow \left\{ \begin{array}{l} \displaystyle \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} (\mathbf{Id} + \hat{\omega}_2) \circ \sigma_2(s) \\ \displaystyle \lim_{s\to\infty} \hat{\sigma}_1(s) > \lim_{s\to\infty} \hat{\alpha}_1(s) \\ \hat{\omega}_1, \hat{\omega}_2 \in \mathcal{K} \end{array} \right. \tag{76}$$

$$\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) \ \Rightarrow \left\{ \begin{array}{l} \displaystyle \lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} (\mathbf{Id} + \hat{\omega}_2) \circ \sigma_2(s) \\ \displaystyle \lim_{s\to\infty} (\mathbf{Id} + \hat{\omega}_1) \circ \hat{\sigma}_1(s) = \lim_{s\to\infty} \hat{\alpha}_1(s) \\ \hat{\omega}_1 \circ \hat{\sigma}_1(s) > 0,\ \hat{\omega}_2 \circ \sigma_2(s) > 0,\ \forall s \in (0, \infty) \end{array} \right. \tag{77}$$

$$\hat{\tau}_i = \mathbf{Id} + \hat{\omega}_i, \quad i = 1, 2\ . \tag{78}$$

Furthermore, the claim can be fulfilled by $\hat{\omega}_1$, $\hat{\omega}_2 \in \mathcal{K}_\infty$ if there exist $\omega_1, \omega_2 \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1^{-1} \circ \alpha_1 \circ (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{79}$$

is satisfied under the assumption of

$$\lim_{s\to\infty}\alpha_2(s)=\infty \quad \vee \quad \lim_{s\to\infty}\alpha_2(s)>\lim_{s\to\infty}\sigma_2(s)\ . \tag{80}$$

Using the functions given in Lemma 3 and $L := \lim_{s\to\infty}\hat{\sigma}_1(s)$, we define continuous functions $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ as

$$\begin{aligned}\lambda_1(s) &:= \left[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1 \circ\bar{\alpha}_1^{-1}(s)\right]\left[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \\ &\qquad\cdot\left[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \qquad (81)\\ \lambda_2(s) &:= \hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\left[\nu\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\right] \cdot\left[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\right], \qquad (82)\end{aligned}$$

where δ_i and τ_1 are any class $\mathcal{K}_\infty$ functions satisfying

$$\mathbf{Id}-\delta_i\in\mathcal{K}_\infty,\quad i=1,2 \tag{83}$$

$$\tau_1=\mathbf{Id}+k\hat{\omega}_1 \tag{84}$$

for some $k\in(0,1)$, and $\nu,\psi:(0,L)\to\mathbb{R}_+$ are any continuous functions which satisfy

$$0<\nu(s)<\infty,\ 0<\psi(s)<\infty,\quad \forall s\in(0,L) \tag{85}$$

and fulfill

$$\begin{aligned}&\left[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\right] \cdot\left[\nu\circ\hat{\sigma}_1(s)\right]\left[\psi\circ\hat{\sigma}_1(s)\right] \quad :\text{non-decreasing} \qquad (86)\\ &\hat{\sigma}_1(s)\left[\nu\circ\hat{\sigma}_1(s)\right]\left[\psi\circ\hat{\sigma}_1(s)\right] \quad :\text{non-decreasing} \qquad (87)\\ &\left[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \cdot\left[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \quad :\text{non-decreasing} \qquad (88)\\ &\left[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\hat{\tau}_2\circ\sigma_2(s)\right] \cdot\left[\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\hat{\tau}_2\circ\sigma_2(s)\right]\sigma_2(s)\\ &\quad\le\left[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right]\left[\delta_2\circ\hat{\omega}_2\circ\sigma_2(s)\right] \cdot\left[\delta_1\circ k\hat{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right] \qquad (89)\end{aligned}$$

for all $s\in\mathbb{R}_+$. Note that $\tau_1\in\mathcal{K}_\infty$ holds since $s+k\hat{\omega}_1(s)=k(s+\hat{\omega}_1(s))+(1-k)s$ and $\hat{\tau}_1\in\mathcal{K}_\infty$.

The following demonstrates that the pair of $\{\lambda_1,\lambda_2\}$ in (81) and (82) yields a Lyapunov function V_{cl} establishing the 0-GAS, iISS and ISS of the interconnected system Σ under appropriate small-gain conditions.

Theorem 6: Consider $\sigma_{r1},\sigma_{r2}\in\mathcal{P}_0$, a quadruplet $(\alpha_1,\alpha_2,\sigma_1,\sigma_2)$ satisfying (68) and (69), and $V_i:(t,x_i)\in\mathbb{R}_+\times\mathbb{R}^{n_i}\to\mathbb{R}_+$, $i=1,2$, satisfying (20) for some $\underline{\alpha}_i$, $\bar{\alpha}_i\in\mathcal{K}_\infty$. Then, we have the following.

(i) Suppose that $\sigma_{r1}(s)\equiv 0$, $\sigma_{r2}(s)\equiv 0$ and *(M1)* $\vee$ *(M2)* hold. If (70) is satisfied, the functions (81) and (82) satisfy

$$\sum_{i=1}^{2}\lambda_i(V_i(t,x_i))\left\{-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|)\right\} \le\sum_{i=1}^{2}-\alpha_{cl,i}(|x_i|)+\sigma_{cl,i}(|r_i|),$$
$$\forall x_1\in\mathbb{R}^{n_1},x_2\in\mathbb{R}^{n_2},r_1\in\mathbb{R}^{m_1},r_2\in\mathbb{R}^{m_2},t\in\mathbb{R}_+ \tag{90}$$

for some $\alpha_{cl,1},\alpha_{cl,2}\in\mathcal{P}$ and $\sigma_{cl,1}(s)=\sigma_{cl,2}(s)\equiv 0$.

(ii) Suppose that *(J1)* $\vee$ *(J2)* $\vee$ *(J3)* and

$$L<\infty\Rightarrow\ \lim_{s\to L}\nu(s)<\infty,\ \lim_{s\to L}\psi(s)<\infty \tag{91}$$

hold. If there exist $\omega_i\in\mathcal{K}_\infty$, $i=1,2$ such that (79) is satisfied, the functions (81) and (82) with $\hat{\omega}_1$, $\hat{\omega}_2\in\mathcal{K}_\infty$ satisfy (90) for some $\alpha_{cl,1},\alpha_{cl,2}\in\mathcal{K}$ and some $\sigma_{cl,1},\sigma_{cl,2}\in\mathcal{P}_0$ fulfilling

$$\alpha_1,\alpha_2\in\mathcal{K}_\infty\quad\Rightarrow\quad\alpha_{cl,1},\alpha_{cl,2}\in\mathcal{K}_\infty \tag{92}$$

$$\sigma_{r,i}(s)\equiv 0\quad\Rightarrow\quad\sigma_{cl,i}(s)\equiv 0\ \ . \tag{93}$$

There always exist functions ν and ψ fulfilling (85), (86), (87), (88), (89) and (91). The existence and the construction are addressed in Subsection VI-B. The task of finding a pair $\{\lambda_1,\lambda_2\}$ which solves (90) is referred to as a state-dependent scaling problem in [11]. In Theorem 6 *(ii)*, the property (90) resulting in (32) yields the iISS of the interconnected system Σ. Theorem 6 *(i)* demonstrates that the amplification factors ω_1,ω_2 in the small-gain condition (79) can be replaced by a strict inequality sign as far as 0-GAS is concerned. Note that the existence of $\omega_1,\omega_2\in\mathcal{K}_\infty$ achieving (79) implies not only (70), but also *(M1)* $\vee$ *(M2)*.

Remark 11: When ω_1 and ω_2 are restricted to linear functions, the functions in (81) and (82) reduce to the ones given in [11] derived for the ISS case $\alpha_1(\infty)=\alpha_2(\infty)=\infty$ (See also Remark 13). The previous iISS result in [11] not only is limited to linear ω_i's, but also is based on a pair of λ_1 and λ_2 different from the ISS case. There has been a gap between the two Lyapunov functions which deal with iISS and ISS separately. The iISS Lyapunov function given in [11] for linear ω_i's can be obtained from (81)-(82) with

$$\nu(s)=\frac{\tilde{\lambda}_2\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}(s)}{s\psi(s)}\ ,$$

where $\tilde{\lambda}_2$ denotes the function λ_2 derived in [11], and δ_i's are restricted to linear functions. The pair (81)-(82) unifies the treatment of iISS and ISS systems, and includes all solutions given in the previous study [11] and covers non-linear ω_i's. Moreover, Theorem 6 *(i)* shows that the Lyapunov function (28) with (81) and (82) can also establish 0-GAS with ω_i's which are not necessarily positive definite. See Remark 3 for the benefit of (81) and (82) in terms of the less restrictive small-gain condition.

In order to understand the idea of the assumption *(M1)* $\vee$ *(M2)* for 0-GAS and the assumption *(J1)* $\vee$ *(J2)* $\vee$ *(J3)* for iISS, the following lemma is useful.

Lemma 4: Given $\underline{\alpha}_i$, $\bar{\alpha}_i\in\mathcal{K}_\infty$, $i=1,2$, the following propositions hold true for each quadruplet $(\alpha_1,\alpha_2,\sigma_1,\sigma_2)$ satisfying (68), (69):

$$\{\ \textit{(M1)}\vee\textit{(M2)}\ \}\ \Leftrightarrow\ \neg\textit{(M3)}$$
$$\{\ \textit{(J1)}\vee\textit{(J2)}\vee\textit{(J3)}\ \}\ \Leftrightarrow\ \neg\textit{(J4)}\ \ .$$

The case of *(J4)* allows $\infty=\limsup_{s\to\infty}\sigma_{rk}(s)>\lim_{s\to\infty}\alpha_k(s)$. Notice that $\infty=\limsup_{s\to\infty}\sigma_{rk}(s)$ implies the unbounded influence of r_k on Σ_k. In this situation, *(J4)* implies that the underdamped state x_k of Σ_k affects Σ_{3-k} through the unbounded function σ_{3-k}. If the influence of r_k is small enough, we can still obtain iISS of Σ in the case of *(J4)*. In fact, there exists $\epsilon > 0$ for which we can obtain $\alpha_{cl,1}, \alpha_{cl,2} \in \mathcal{K}$ and $\sigma_{cl,1}, \sigma_{cl,2} \in \mathcal{P}_0$ if (79) holds with $\omega_1, \omega_2 \in \mathcal{K}_\infty$ and $\lim_{s\to\infty} \sigma_{rk}(s) \leq \epsilon$.

Remark 12: The situation *(M3)* is referred to as the no gap case in [1] which considers the interconnected system Σ without any external signals r_i, $i = 1, 2$. All situations considered in Theorem 1 of [1] with their small-gain condition are covered by Theorem 6 *(i)*. Thus, this paper gives a new interpretation of the 0-GAS result of [1] in terms of the construction of Lyapunov functions. Due to (86) and (82), the function V_{cl} constructed with λ_1 and λ_2 in the case of

$$\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} = 1 \tag{94}$$

increases toward infinity as x_2 approaches the boundary where $V_2(x_2) = X_2$ holds for $X_2 := \lim_{s\to\infty} \underline{\alpha}_2 \circ \hat{\sigma}_1^{-1} \circ \hat{\alpha}_1(s)$. Indeed, $\lim_{s \nearrow X_2} \lambda_2(s) = \infty$ holds since (94) implies $\hat{\omega}_2(s) = 0$ for $s \in [\sigma_2(\infty), \infty)$. Thus, the function V_{cl} is not qualified as a Lyapunov function of 0-GAS of the interconnected system if X_2 is finite. The assumption $\alpha_2(\infty) = \sigma_2(\infty)$ in *(M1)* guarantees $X_2 = \infty$. The information about gain is not enough to discriminate between stable and unstable behavior in the no gap case as discussed in [1]. Indeed, the system given by (38)-(42) on $\mathbb{R}_+^2$ satisfies *(M3)*. Although the small-gain condition (70) is fulfilled, there exist unbounded trajectories for $0 < p < 1$. This fact justifies the assumption of *(M1)* $\vee$ *(M2)* in Theorem 6 *(i)*, and this paper gives a Lyapunov interpretation to the no gap case.

B. Construction of ψ

Once a function ψ satisfying (85), (89) and (91), is given, we can always select a function ν required in Theorem 6 straightforwardly. Such a desired function ψ is constructed as follows: First, define

$$Q(t) = \begin{cases} \dfrac{1}{m(t)-t}\left(\dfrac{\hat{d}(t)}{\hat{b}(t)}-1\right) & , \quad t \in (0, S) \\ \dfrac{1}{m(S)-S}\left(\limsup_{s\to S} \dfrac{\hat{d}(s)}{\hat{b}(s)}-1\right) & , \quad t \in [S, R) \end{cases} \tag{95}$$

$$\hat{b}(s) = b \circ \eta^{-1}(s), \ \hat{d}(s) = d \circ \eta^{-1}(s) \tag{96}$$

$$m(s) = \tau_1^{-1} \circ \alpha \circ \eta^{-1}(s) \tag{97}$$

$$S = \lim_{s\to\infty} \eta(s), \quad R = \lim_{s\to\infty} \tau_1^{-1} \circ \alpha(s)$$

for a real number $k \in (0, 1)$, where

$$\alpha(s) = \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)$$
$$b(s) = \left[\delta_1 \circ k\hat{\omega}_1 \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] \left[\delta_2 \circ \hat{\omega}_2 \circ \sigma_2(s)\right]$$
$$d(s) = \left[\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \hat{\tau}_2 \circ \sigma_2(s)\right] \sigma_2(s)$$
$$\eta(s) = \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \hat{\tau}_2 \circ \sigma_2(s) \ .$$

We can always pick a non-decreasing function $\bar{Q} : (0, R) \to \mathbb{R}$ satisfying

$$\bar{Q}(s) \geq \max\{Q(s), 0\}, \quad \forall s \in (0, R) \ . \tag{98}$$

In the case of $\limsup_{s\to 0} Q(s) = \infty$, let $\bar{Q}$ be of the form

$$\bar{Q}(s) = \frac{1}{\int_0^s \xi(r)dr}, \quad s \in (0, R) \ , \tag{99}$$

and we can pick a function $\xi : (0, R) \to \mathbb{R}$ satisfying

$$\xi(s) \leq 1, \ 0 < \int_0^s \xi(r)dr \leq \frac{1}{\max\{Q(s), 0\}} \quad \forall s \in (0, R) \ . \tag{100}$$

Then, for arbitrary $C > 0$ and $T \in (0, R)$, define ψ by

$$\psi(s) = Ce^{\int_T^s \bar{Q}(t)dt}, \quad s \in (0, R) \tag{101}$$

$$\psi(s) = \psi(R), \qquad s \in [R, \infty) \ . \tag{102}$$

Note that (71) implies $S \leq R$. It can be verified that the above function ψ satisfies

$$0 < \psi(s) < \infty, \quad \forall s \in (0, \infty) \tag{103}$$

$$[\psi \circ \eta(s)]\, d(s) \leq \left[\psi \circ \tau_1^{-1} \circ \alpha(s)\right] b(s), \ \forall s \in (0, \infty) \ . \tag{104}$$

The inequality (104) corresponds to (89). The property (103) ensures (85) and (91) in terms of ψ.

It is stressed that, when supply rates for Σ_i, $i = 1, 2$ are given by

$$-\alpha_i(V_i(t, x_i)) + \sigma_i(V_{3-i}(t, x_{3-i})) + \sigma_{ri}(|r_i|)$$

instead of $-\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|)$, all developments in this Section VI remain valid by replacing $\underline{\alpha}_i$ and $\bar{\alpha}_i$ with the identity map, and replacing $|x_i|$ with V_i.

Remark 13: If $Q(s) \leq 0$ holds for all $s \in (0, R)$, the choice $\bar{Q}(s) = 0$ fulfills (98), which yields $\psi(s) = C > 0$. If there exists $K \in (-\infty, 0) \cup [1, \infty)$ such that

$$\sup_{t\in(0,R)} t\, Q(t) \leq K \tag{105}$$

holds, the choice $\bar{Q}(s) = K/s$ yields $\psi(s) = Cs^K$. In the case of uniform contraction where ω_1 and ω_2 are linear, there exist a sufficiently large $K < \infty$ such that (105) holds. When we take $\psi(s) = Cs^K$, the functions λ_1 and λ_2 reduce to the ones used in earlier results [11] dealing with uniformly contractive loop gain for ISS systems.

VII. CONCLUSIONS

This paper has proved that the nonlinear small-gain-type condition is a necessary and sufficient stability criterion for the stability of a family of interconnected systems consisting of iISS subsystems. Both the necessity and the sufficiency have been investigated from a Lyapunov perspective. A $\mathbf{C}^1$ Lyapunov function can be explicitly constructed whenever the small-gain condition is satisfied. This paper has derived a common formula of Lyapunov functions applicable equally to iISS and ISS systems.

We have allowed the subsystems to be unspecified so that the exact information of their ODE models are not assumed. Instead, the subsystems are supposed to belong to sets defined by dissipation inequalities of the iISS type, which is conformable to the idea of modeling uncertainty. This paper has proved that the interconnection of two unspecified iISS subsystems is guaranteed to be stable only if at least one of the two subsystems is ISS with respect to the feedback input. Another important result is that the nonlinear small-gain condition is necessary for the stability of the family of interconnected uncertain systems described by the supply rates. Nevertheless, it is worth noting that, when ODEs of the two merely iISS subsystems are known, an interconnection of them may remain stable. In the theory of linear robust control, the necessity of $\mathcal{L}_p$ small-gain conditions still holds for the interconnection of a completely known subsystem and an uncertain subsystem. This paper claims that the necessity for such a partially known system is fragile in the nonlinear case.

Finally, we notice that this paper does not address the necessity of the class $\mathcal{K}_\infty$ property of ω_i's for achieving the iISS property of the whole interconnected system. Identifying some necessary conditions which are more restrictive than those in the uniform 0-GAS case but less restrictive than those in the ISS case remains to be an interesting subject of future study.

Appendix A
Proof of Lemma 1

By assumption, there exist $N > 1$ and $J > 0$ such that $\alpha\in\mathcal{P}$ and $\sigma\in\mathcal{K}$ are written in the form of

$$\alpha(|x|) = \hat{\alpha}(|x|)|x|^N, \quad \sigma(|u|) = \hat{\sigma}(|u|)|u|^J$$

with some functions $\hat{\alpha}(s)$ and $\hat{\sigma}(s)$ which are continuous on $[0,\infty)$. The class $\mathbf{C}^1$ property of α and σ also implies that $\hat{\alpha}$ and $\hat{\sigma}$ are $\mathbf{C}^1$ in $(0,\infty)$. Pick a real number $Q \geq 1$ so that

$$\frac{1}{N} + \frac{1}{JQ} < 1$$

is satisfied. Let $\epsilon \in (0, \bar{\epsilon}]$. In the case of $(1/N) + (1/J) < 1$, let $Q = 1$ and $\epsilon = 0$. Define $\theta \in \mathcal{K}_\infty$ as

$$\theta(s) = \begin{cases} \sigma(\epsilon)\left(s/\sigma(\epsilon)\right)^Q & , \quad \text{for } s \in [0, \sigma(\epsilon)) \\ s & , \quad \text{for } s \in [\sigma(\epsilon), \infty) \ . \end{cases}$$

The class $\mathcal{K}$ function $\tilde{\sigma}$ given by $\tilde{\sigma}(s) = \theta \circ \sigma(s)$ satisfies

$$\begin{array}{lll} \tilde{\sigma}(s) = \sigma(s) = 0 & , & s = 0 \\ \tilde{\sigma}(s) < \sigma(s) & , & \forall s \in (0, \epsilon) \\ \tilde{\sigma}(s) = \sigma(s) & , & \forall s \in [\epsilon, \infty) \ . \end{array} \tag{106}$$

Define $p > 1$ by

$$\frac{1}{p} = 1 - \frac{1}{JQ} \quad . \tag{107}$$

Let $q = JQ$ so that $(1/p) + (1/q) = 1$ holds. Define

$$V(x) = \underline{\alpha}(|x|) = \bar{\alpha}(|x|) = |x|^{N/p} \tag{108}$$

$$f(x,u) = f_A(x) + f_B(x,u) \tag{109}$$

$$f_A = \frac{-\mu p}{N}\hat{\alpha}(|x|)|x|^{N/q}x, \quad \mu = \frac{q}{p}(1+\delta) + 1 \tag{110}$$

$$f_B = \frac{p}{N}(q(1+\delta)\hat{\alpha}(|x|))^{1/p}(q\tilde{\sigma}(|u|))^{1/q}x \ . \tag{111}$$

Then, we have

$$\begin{aligned} \frac{\partial V}{\partial x} f \ &= \ \frac{N}{p}|x|^{\frac{N}{p}-2}x^T f \\ &= \ -\mu\hat{\alpha}(|x|)|x|^N \\ &\quad + \left(p\frac{q}{p}(1+\delta)\hat{\alpha}(|x|)|x|^N\right)^{1/p}(q\tilde{\sigma}(|u|))^{1/q} \ . \end{aligned}$$

Applying Young's inequality to the right-hand side, we obtain

$$\begin{aligned} \frac{\partial V}{\partial x} f \ &\leq \ -\left(\mu - \frac{q}{p}(1+\delta)\right)\hat{\alpha}(|x|)|x|^N + \tilde{\sigma}(|u|) \\ &\leq \ -\alpha(|x|) + \sigma(|u|) \end{aligned}$$

Since $q(1+\delta) - \mu = \delta$ holds, we arrive at

$$\begin{aligned} &(1+\delta)\alpha(|x|) = \tilde{\sigma}(|u|) \ \Rightarrow \\ &\qquad \frac{\partial V}{\partial x} f = -\mu\alpha(|x|) + q\tilde{\sigma}(|u|) \\ &\qquad\qquad = (q(1+\delta) - \mu)\alpha(|x|) = \delta\alpha(|x|) \\ &(1+\delta)\alpha(|x|) < \tilde{\sigma}(|u|) \ \Rightarrow \\ &\qquad \frac{\partial V}{\partial x} f > (q(1+\delta) - \mu)\alpha(|x|) = \delta\alpha(|x|) \ . \end{aligned}$$

Thus, we have (46) by virtue of (106). The choice (107) of p implies $N/p > 1$, so that V given by (108) is $\mathbf{C}^1$. The function f_A is Lipschitz at each point in $\mathbb{R}^n$ due to $N/q \geq 0$ and the class $\mathbf{C}^1$ property of $\hat{\alpha}$ on $(0,\infty)$. The function f_B is also locally Lipschitz in x on $\mathbb{R}^n$ since $\hat{\alpha}(s)^{1/p}$ is $\mathbf{C}^1$ on $(0,\infty)$ and bounded on $\mathbb{R}_+$. To verify the local Lipschitzness with respect to $u \in \mathbb{R}^m$, we first obtain $JQ = q$ from (107). Next,

$$\tilde{\sigma}(s)^{1/q} = \sigma(\epsilon)^{1/q}(\hat{\sigma}(s)/\sigma(\epsilon))^{Q/q}|s| \quad , \quad \forall s \in [0, \epsilon]$$

follows from $\sigma \in \mathcal{O}(>0)$. This function $\tilde{\sigma}(s)^{1/q}$ is continuously differentiable in the interval $(0, \epsilon]$ since $\hat{\sigma}(s)^{Q/q}$ is class $\mathbf{C}^1$. The function $\tilde{\sigma}(s)^{1/q}$ is also Lipschitz at zero since $Q/q > 0$. The identity

$$\tilde{\sigma}(s)^{1/q} = \hat{\sigma}(s)^{1/q}|s|^{J/q} \quad , \quad \forall s \in [\epsilon, \infty)$$

together with $q > 1$ and $J > 0$ guarantees that $\tilde{\sigma}(s)^{1/q}$ is $\mathbf{C}^1$ at each $s \in [\epsilon, \infty)$ due to the continuous differentiability of $\hat{\sigma}(s)^{1/q}$. Hence, the function f_B is locally Lipschitz at all $u \in \mathbb{R}^m$.

Appendix B
Proof of Theorem 4

We first deal with S_1 and S_2 given by (48) and we begin with proving *(ii)*.

(ii) Suppose that (51) is not satisfied for each $i = 1, 2$. This assumption is equivalent to

$$\begin{aligned} &\liminf_{s\to\infty} \alpha_i(s) < \infty \ \wedge \\ &\qquad \liminf_{s\to\infty} \alpha_i(s) < \lim_{s\to\infty} \sigma_i(s) + \sup_{s\in\mathbb{R}_+} \sigma_{ri}(s) \end{aligned}$$

for $i = 1, 2$. Due to $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{P}_0$, there exist $v_i > 0$, $w_i > 0$ and $\delta_i > 0$ for $i = 1, 2$ such that

$$\begin{aligned} (1+\delta_1)\alpha_1(s) < \sigma_1(w_2) + \sigma_{r1}(v_1), \quad \forall s \in \{h_{11}, h_{12}, ...\} \\ (1+\delta_2)\alpha_2(s) < \sigma_2(w_1) + \sigma_{r2}(v_2), \quad \forall s \in \{h_{21}, h_{22}, ...\} \end{aligned}$$

hold for some increasing sequences $h_{1n} \to \infty$ and $h_{2n} \to \infty$ satisfying h_{11}, $h_{21} \geq 0$, respectively. For all integers j and k satisfying $h_{1j} \geq w_1$ and $h_{2k} \geq w_2$, the properties

$$\begin{aligned} &|x_1| = h_{1j}, \ |x_2| \geq h_{2k} \Rightarrow \\ &\qquad (1+\delta_1)\alpha_1(|x_1|) < \sigma_1(|x_2|) + \sigma_{r1}(|r_1|) \\ &|x_1| \geq h_{1j}, \ |x_2| = h_{2k} \Rightarrow \\ &\qquad (1+\delta_2)\alpha_2(|x_2|) < \sigma_2(|x_1|) + \sigma_{r2}(|r_2|) \end{aligned}$$

hold as long as r_1 and r_2 satisfy $|r_1| \geq v_1, |r_2| \geq v_2$. Lemma 1 with replacement of $\sigma(|u|)$ with $\sigma_i(|u_i|)+\sigma_{ri}(|r_i|)$ guarantees the existence of $f_1(x_1, u_1, r_1)$, $f_2(x_2, u_2, r_2)$: $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{k_i} \to \mathbb{R}$, $\mathbf{C}^1$ functions V_1, V_2: $\mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_1$, $\bar{\alpha}_1$, $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and

$$\begin{aligned} &\underline{\alpha}_i(|x_i|) = V_i(x_i) = \bar{\alpha}_i(|x_i|) \qquad (112)\\ &(1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|) + \sigma_{ri}(|r_i|) \ \Rightarrow \\ &\qquad\qquad \frac{\partial V_i}{\partial x_i} f_i > \delta_i \alpha_i(|x_i|) \end{aligned}$$

hold for $i = 1, 2$. These systems Σ_1 and Σ_2 satisfy

$$|x_1| = h_{1j},\ |x_2| \geq h_{2k} \Rightarrow \frac{\partial V_1}{\partial x_1} f_1 > \delta_i \alpha_1(|x_1|) \quad (113)$$

$$|x_1| \geq h_{1j},\ |x_2| = h_{2k} \Rightarrow \frac{\partial V_2}{\partial x_2} f_2 > \delta_i \alpha_2(|x_2|) \quad (114)$$

for all $|r_1| \geq v_1, |r_2| \geq v_2$. Define

$$\begin{aligned} \mathbf{U}(l_1, l_2) = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ V_i(x_i) \geq \bar{\alpha}_i(l_i), i = 1, 2\} \ .(115) \end{aligned}$$

Due to (112), the pair of (113) and (114) implies that trajectories starting from $(x_1(0), x_2(0)) \in \mathbf{U}(h_{1j}, h_{2k})$ stay in $\mathbf{U}(h_{1j}, h_{2k})$ forever if $|r_1| = v_1$ and $|r_2| = v_2$ hold for all $t \in \mathbb{R}_+$. The trajectories remain in $\mathbf{U}(h_{1j}, h_{2k})$ for the same r_1 and r_2 however large h_{1j} and h_{2k} are. This invariance property implies that the interconnected system does not have finite gain in terms of ISS [22].

(i) Suppose that (50) does not hold for $i = 1, 2$. There exist $w_i, \delta_i > 0$ for $i = 1, 2$ such that

$$\begin{aligned} (1+\delta_1)\alpha_1(s) < \sigma_1(w_2), \quad \forall s \in \{h_{11}, h_{12}, ...\} \\ (1+\delta_2)\alpha_2(s) < \sigma_2(w_1), \quad \forall s \in \{h_{21}, h_{22}, ...\} \end{aligned}$$

are satisfied for some increasing sequences $h_{1n} \to \infty$ and $h_{2n} \to \infty$ satisfying h_{11}, $h_{21} \geq 0$, respectively. Lemma 1 guarantees the existence of $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$, $i = 1, 2$, such that (113) and (114) hold. Trajectories starting from $\mathbf{U}(h_{1j}, h_{2k})$ remain in $\mathbf{U}(h_{1j}, h_{2k})$ for arbitrary h_{1j} and h_{2k}. Therefore, the interconnection is not 0-GAS.

In the case of (49), by assumption there exist $M_i > 1$ and $L_i > 0$ such that $\alpha_i \in \mathcal{O}(M_i)$ and $\sigma_i \in \mathcal{O}(L_i)$ hold for $i = 1, 2$. Define $\breve{\alpha}_i = \alpha_i(s^{K_i})$ and $\breve{\sigma}_i = \sigma_i(s^{K_{3-i}})$ for some $K_i > 1$, $i = 1, 2$. Then, there exist continuous functions $\hat{\alpha}_i, \hat{\sigma}_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} \breve{\alpha}_i(|x_i|) = \hat{\alpha}_i(|x_i|)|x_i|^{N_i},\ N_i = K_i M_i > 1 \\ \breve{\sigma}_i(|x_{3-i}|) = \hat{\sigma}_i(|x_{3-i}|)|x_{3-i}|^{J_i},\ J_i = K_{3-i} L_i > 0 \end{aligned}$$

hold for $i = 1, 2$. Since α_i and σ_i are $\mathbf{C}^1$, the functions $\hat{\alpha}_i$ and $\hat{\sigma}_i$ are also $\mathbf{C}^1$ on $(0, \infty)$. Lemma 1 yields a Lipschitz continuous time-invariant system $\Sigma_i \in \mathcal{S}_i(n_i, \breve{\alpha}_i, \breve{\sigma}_i, \sigma_{ri})$ with $V_i(x_i) = |x_i|^{K_i}$ for each $i = 1, 2$. The property $\mathcal{S}_i(n_i, \breve{\alpha}_i, \breve{\sigma}_i, \sigma_{ri}) = \mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ completes the proof.

Appendix C
Proof of Theorem 5

The following deals with (48). The technique to deal with (49) is the same as Theorem 4.

(i): Assume $\alpha_i \in \mathcal{K}$ temporarily and let $\tilde{\alpha}_i = \alpha_i$, $i = 1, 2$. Suppose that there exists $l_1 \in (0, \infty)$ such that

$$\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(l_1) \geq \alpha_1(l_1) \quad (116)$$

holds. Pick $l_2 \in (0, \infty)$ so that $\alpha_2^{-1} \circ \sigma_2(l_1) \geq l_2 \geq \sigma_1^{-1} \circ \alpha_1(l_1)$ is satisfied. Using $\alpha_2, \sigma_1 \in \mathcal{K}$, we obtain $\alpha_2(l_2) \leq \sigma_2(l_1)$ and $\alpha_1(l_1) \leq \sigma_1(l_2)$. Suppose $|r_1(t)| = |r_2(t)| = 0$ for all $t \in \mathbb{R}_+$. Lemma 1 guarantees the existence of two time-invariant systems $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_2, \sigma_{r2})$ achieving (112) and

$$\alpha_i(|x_i|) \leq \sigma_i(|x_{3-i}|) \Rightarrow \frac{\partial V_i}{\partial x_i} f_i \geq 0$$

for $i = 1, 2$. This leads to the following:

$$|x_1| = l_1,\ |x_2| \geq l_2 \Rightarrow \frac{\partial V_1}{\partial x_1} f_1 \geq 0 \quad (117)$$

$$|x_1| \geq l_1,\ |x_2| = l_2 \Rightarrow \frac{\partial V_2}{\partial x_2} f_2 \geq 0\ . \quad (118)$$

Define $\mathbf{U}(l_1, l_2)$ as in (115). Due to (112), the property characterized by (117) and (118) implies that trajectories starting from $x(0) \in \mathbf{U}(l_1, l_2)$ remain in $\mathbf{U}(l_1, l_2)$. This invariance contradicts the 0-GAS. Next, consider the case of $\alpha_i \in \mathcal{P} \setminus \mathcal{K}$. Suppose that

$$\begin{aligned} &\alpha_{i.1},\ \alpha_{i.2} \in \mathcal{K}, \quad i = 1, 2 \\ &\alpha_{i.1}(s) \geq \alpha_{i.2}(s),\ \forall s \in \mathbb{R}_+, \quad i = 1, 2 \end{aligned}$$

hold. Then, if there exists $l_1 \in (0, \infty)$ such that

$$\sigma_1 \circ \alpha_{2.k}^{-1} \circ \sigma_2(l_1) \geq \alpha_{1.k}(l_1) \quad (119)$$

holds for $k = 1$, the same l_1 also satisfies (119) for $k = 2$. This property implies that the negation of (53) implies the existence of $l_1 \in (0, \infty)$ satisfying

$$\sigma_1 \circ \tilde{\alpha}_2^{-1} \circ \sigma_2(l_1) \geq \tilde{\alpha}_1(l_1) \quad (120)$$

for all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (54). Define $l_2 \in (0, \infty)$ satisfying $\tilde{\alpha}_2^{-1} \circ \sigma_2(l_1) \geq l_2 \geq \sigma_1^{-1} \circ \tilde{\alpha}_1(l_1)$. If

$$\tilde{\alpha}_i(l_i) = \alpha_i(l_i), \quad i = 1, 2 \quad (121)$$

holds, the argument given above for $\alpha_i \in \mathcal{K}$, $i = 1, 2$ leads to the existence of a pair of systems whose interconnection is not 0-GAS. Suppose that there exists $l_1 \in (0, \infty)$ such that

$$\tilde{\alpha}_1(l_1) < \alpha_1(l_1) \quad (122)$$

and (120) hold for "all" $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (54). Then, $\tilde{\alpha}_1(l_1) \leq \sigma_1(l_2)$ holds with any $\tilde{\alpha}_2 \in \mathcal{K}$ fulfilling (54), which implies that there exists $\bar{l}_1 \in (l_1, \infty)$ such that $\alpha_1(\bar{l}_1) \leq \sigma_1(l_2)$ holds. If (120) and

$$\tilde{\alpha}_2(l_2) < \alpha_2(l_2) \quad (123)$$

hold for all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (54), there exists $\bar{l}_2 \in (l_2, \infty)$ such that $\alpha_2(\bar{l}_2) \leq \sigma_2(l_1)$ holds. If (122) and (123) are satisfied simultaneously, we have $\alpha_1(\bar{l}_1) \leq \sigma_1(\bar{l}_2)$ and $\alpha_2(\bar{l}_2) \leq \sigma_2(\bar{l}_1)$. Hence, the rest of the proof is the same as the case of $\alpha_i \in \mathcal{K}$, $i = 1, 2$.

(ii): Consider the case of $\sigma_{r1}, \sigma_{r2} \in \mathcal{K}_\infty$. In order to prove the claim by contradiction, assume that (56) is violated for all pairs of $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$. First, suppose that there

exists $l_1 \in (0,\infty)$ such that (116) holds with all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (54). Then, the claim *(i)* proves that $x=0$ is not guaranteed to be GAS, which implies that the interconnection is not ISS. Next, we suppose that there are no $l_1 \in (0,\infty)$ and no $\tilde{\alpha}_i \in \mathcal{K}$ satisfying (116) and (54). Since all pair of $\omega_i \in \mathcal{K}_\infty$, $i=1,2$ violate (56), there exist continuous functions $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a non-empty set $\mathbf{Y}$ such that

$$(\mathbf{Id}+\omega_1)\circ\sigma_1\circ\tilde{\alpha}_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(s)=\tilde{\alpha}_1(s), \quad \forall s\in\mathbb{R}_+ \tag{124}$$

$$\mathbf{Id}+\omega_1, \mathbf{Id}+\omega_2 \in \mathcal{K}_\infty$$

$$\lim_{s\to\infty}\omega_j(s)<\infty, \quad \forall j\in\mathbf{Y}\subset\{1,2\} \tag{125}$$

are satisfied for some $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (54). The property (125) yields

$$\lim_{s\to\infty}\omega_j\circ\sigma_j(s)<\infty\ . \tag{126}$$

Since σ_{r1} and σ_{r2} are of class $\mathcal{K}_\infty$, there exists $R_j \in (0,\infty)$ such that

$$\lim_{s\to\infty}\omega_j\circ\sigma_j(s)\le\sigma_{rj}(|r_j|),\ \forall s\in\mathbb{R}_+,\ \forall|r_j|\ge R_j \tag{127}$$

holds for all $j\in\mathbf{Y}$. Let l_1 be a real number in $(0,\infty)$, which is now given arbitrarily in contrast to the *(i)* case. Define $l_2(l_1)=\tilde{\alpha}_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(l_1)$ which is of class $\mathcal{K}$. Due to (124), we have $\tilde{\alpha}_2(l_2(l_1))=(\mathbf{Id}+\omega_2)\circ\sigma_2(l_1)$ and $\tilde{\alpha}_1(l_1)=(\mathbf{Id}+\omega_1)\circ\sigma_1(l_2(l_1))$. By replacing σ with $\sigma_i+\sigma_{ri}$ in Lemma 1, we obtain $\Sigma_i\in\mathcal{S}_i(n_i,\tilde{\alpha}_i,\sigma_i,\sigma_{ri})$, such that (112) and

$$\tilde{\alpha}_i(|x_i|)\le\sigma_i(|x_{3-i}|)+\sigma_{ri}(|r_i|)\Rightarrow\frac{\partial V_i}{\partial x_i}f_i\ge 0$$

hold for $i=1,2$. This leads to (117) and (118) for all $|r_j|\ge R_j$, $j\in\mathbf{Y}$ and $r_k=0$, $k\in\{1,2\}\setminus\mathbf{Y}$ since we have (127). Define $\mathbf{U}(l_1,l_2)$ as in (115). The inequalities (117) and (118) imply that trajectories starting from $\mathbf{U}(l_1,l_2)$ remain in $\mathbf{U}(l_1,l_2)$ as long as $|r_j(t)|\ge R_j$ and $r_k(t)=0$ hold. Recall that l_1 is arbitrary in $(0,\infty)$, and independent of R_j. The trajectories for the fixed input $|r_j(t)|=R_j<\infty$ does not leave $\mathbf{U}(l_1,l_2)$ no matter how large l_1 is. This violates the ISS property [22]. Therefore, the interconnected system Σ is not ISS when (56) is violated for all pair of $\omega_i\in\mathcal{K}_\infty$ and all $\tilde{\alpha}_i\in\mathcal{K}$ fulfilling (54), $i=1,2$. Note that $\alpha_i\in\mathcal{P}\setminus\mathcal{K}$ can be handled as in *(i)*. In the case of $\sigma_{ri}\notin\mathcal{K}_\infty$, use $\omega_i(s)\equiv 0$ and $r_i(t)\equiv 0$. The property $\tilde{\alpha}_i(l_i)\le\sigma_i(l_{3-i})+\sigma_{ri}(|r_i|)$ is replaced by $\tilde{\alpha}_i(l_i)\le\sigma_i(l_{3-i})$.

Appendix D
Proof of Theorem 6

(ii): The logical sum of *(J1)*, *(J2)*, *(J3)* is equivalent to the logical sum of

(N1) $\lim_{s\to\infty}\alpha_2(s)=\infty \quad \wedge \quad \{\lim_{s\to\infty}\alpha_1(s)=\infty \ \vee\ \lim_{s\to\infty}\sigma_2(s)<\infty\}$

(N2) $\lim_{s\to\infty}\alpha_2(s)<\infty \ \wedge\ \lim_{s\to\infty}\sigma_1(s)<\infty$

under the assumption (80). We first prove the claim in the case of *(N1)*. For notational simplicity, we use the following notations:

$$\underline{\omega}_1=k\hat{\omega}_1,\quad \underline{\omega}_2=\hat{\omega}_2,\quad \tau_2=\hat{\tau}_2,\quad \hat{\alpha}_2=\alpha_2,\quad \hat{\sigma}_2=\sigma_2$$

Replace σ_{ri} by $\bar{\sigma}_{ri}\in\mathcal{K}$ satisfying $\sigma_{ri}(s)\le\bar{\sigma}_{ri}(s)$ for all $s\in\mathbb{R}_+$, $i=1,2$. Due to (83), we can pick a class $\mathcal{K}_\infty$ function τ_{ri} fulfilling

$$\underline{\omega}_i\circ\tau_i^{-1}-\delta_i\circ\underline{\omega}_i\circ\tau_i^{-1}-\tau_{ri}^{-1}\in\mathcal{K}_\infty$$

for each $i=1,2$. The rest of the proof does not involve $\bar{\sigma}_{ri}$ and τ_{ri} if $\sigma_{ri}(r_i)$ is identically zero. Define

$$\theta_i(s)=\bar{\alpha}_i\circ\hat{\alpha}_i^{-1}\circ\tau_i\circ\hat{\sigma}_i(s),\quad s\in[0,Y_i) \tag{128}$$

$$\theta_{ri}(s)=\bar{\alpha}_i\circ\hat{\alpha}_i^{-1}\circ\tau_{ri}\circ\bar{\sigma}_{ri}(s),\quad s\in[0,Y_{ri})$$

$$Y_1=\lim_{s\to\infty}\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1(s)$$

$$Y_{r1}=\left\{\begin{array}{ll}\infty & \text{, if } \lim_{s\to\infty}\hat{\alpha}_1(s)\ge\lim_{s\to\infty}\tau_{r1}\circ\bar{\sigma}_{r1}\\ \lim_{s\to\infty}\bar{\sigma}_{r1}^{-1}\circ\tau_{r1}^{-1}\circ\hat{\alpha}_1(s) & \text{, otherwise}\end{array}\right.$$

$$Y_2=\infty,\quad Y_{r2}=\infty$$

for $i=1,2$. The function $\lambda_1(s)$ given by (81) satisfies $\lambda_1(s)>0$ for all $s\in(0,\infty)$ and it is non-decreasing on $\mathbb{R}_+$ since (85) and (86). Define non-decreasing functions $\lambda_{\theta1},\lambda_{\theta r1}:\mathbb{R}_+\to\mathbb{R}_+$ by

$$\lambda_{\theta1}(s)=\left\{\begin{array}{lll}\lambda_1\circ\theta_1(s) & , & s\in[0,Y_1)\\ \lim_{s\to\infty}\lambda_1(s) & , & s\in[Y_1,\infty)\end{array}\right. \tag{129}$$

$$\lambda_{\theta r1}(s)=\left\{\begin{array}{lll}\lambda_1\circ\theta_{r1}(s) & , & s\in[0,Y_{r1})\\ \lim_{s\to\infty}\lambda_1(s) & , & s\in[Y_{r1},\infty)\ .\end{array}\right. \tag{130}$$

By virtue of (73), $\hat{\sigma}_1(\infty)>\tau_1^{-1}\circ\hat{\alpha}_1(\infty)$ holds if and only if $\hat{\alpha}_1(\infty)<\infty$. Thus,

$$Y_1<\infty\vee Y_{r1}<\infty\ \Rightarrow\ \lim_{s\to\infty}\hat{\alpha}_1(s)<\infty\ \Rightarrow\ \lim_{s\to\infty}\lambda_1(s)<\infty \tag{131}$$

follows from (81). The function $\lambda_2(s)$ given by (82) is a non-decreasing function satisfying $\lambda_2(s)>0$ for all $s\in(0,\infty)$ under (85) and (87). Define non-decreasing functions $\lambda_{\theta2},\lambda_{\theta r2}:\mathbb{R}_+\to\mathbb{R}_+$ by

$$\lambda_{\theta2}(s)=\lambda_2\circ\theta_2(s),\ \lambda_{\theta r2}(s)=\lambda_2\circ\theta_{r2}(s)\ ,\ s\in\mathbb{R}_+\ .$$

We obtain

$$\begin{aligned}&\lambda_1(V_1)\{-\hat{\alpha}_1(|x_1|)+\hat{\sigma}_1(|x_2|)+\sigma_{r1}(|r_1|)\}\\ &\quad\le-\lambda_1(\underline{\alpha}_1(|x_1|))\left[\underline{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1(|x_1|)-\tau_{r1}^{-1}\circ\hat{\alpha}_1(|x_1|)\right]\\ &\qquad+\lambda_{\theta1}(|x_2|)\hat{\sigma}_1(|x_2|)+\lambda_{\theta r1}(|r_1|)\bar{\sigma}_{r1}(|r_1|)\end{aligned} \tag{132}$$

$$\begin{aligned}&\lambda_2(V_2)\{-\hat{\alpha}_2(|x_2|)+\hat{\sigma}_2(|x_1|)+\sigma_{r2}(|r_2|)\}\\ &\quad\le-\lambda_2(\underline{\alpha}_2(|x_2|))\left[\underline{\omega}_2\circ\tau_2^{-1}\circ\hat{\alpha}_2(|x_2|)-\tau_{r2}^{-1}\circ\hat{\alpha}_2(|x_2|)\right]\\ &\qquad+\lambda_{\theta2}(|x_1|)\hat{\sigma}_2(|x_1|)+\lambda_{\theta r2}(|r_2|)\bar{\sigma}_{r2}(|r_2|)\end{aligned} \tag{133}$$

by combining calculations in individual cases divided by $\hat{\alpha}_i(|x_i|)\ge\tau_i\circ\hat{\sigma}_i(|x_{3-i}|)$, $\hat{\alpha}_i(|x_i|)<\tau_i\circ\hat{\sigma}_i(|x_{3-i}|)$, $\hat{\alpha}_i(|x_i|)\ge\tau_{ri}\circ\bar{\sigma}_{ri}(|r_i|)$ and $\hat{\alpha}_i(|x_i|)<\tau_{ri}\circ\bar{\sigma}_{ri}(|r_i|)$. Thus, the inequality (90) is fulfilled with

$$\begin{aligned}\alpha_{cl,i}(s)=&\lambda_i(\underline{\alpha}_i(s))\left[\underline{\omega}_i\circ\tau_i^{-1}\circ\hat{\alpha}_i\right.\\ &\left.-\delta_i\circ\underline{\omega}_i\circ\tau_i^{-1}\circ\hat{\alpha}_i\circ\bar{\alpha}_i^{-1}\circ\underline{\alpha}_i-\tau_{ri}^{-1}\circ\hat{\alpha}_i\right]\end{aligned} \tag{134}$$

$$\sigma_{cl,i}(s)=\lambda_{\theta ri}(|s|)\bar{\sigma}_{ri}(|s|) \tag{135}$$

if λ_1 and λ_2 satisfy

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) \leq \lambda_2(\underline{\alpha}_2(s))\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\right] \quad (136)$$

$$\lambda_{\theta 2}(s)\hat{\sigma}_2(s) \leq \lambda_1(\underline{\alpha}_1(s))\left[\delta_1\circ\underline{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right] \quad (137)$$

for all $s \in \mathbb{R}_+$. Here, $\delta_i \circ \underline{\omega}_i \in \mathcal{K}$ is used. Consider the following three conditions.

$$\sigma_2(s)\left[\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\theta_2(s)\right]\left[\lambda_{\theta 1}\circ\underline{\alpha}_2^{-1}\circ\theta_2(s)\right] \leq \lambda_1(\underline{\alpha}_1(s))\left[\delta_2\circ\underline{\omega}_2\circ\sigma_2(s)\right] \cdot\left[\delta_1\circ\underline{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right],\ s\in\mathbb{R}_+ \quad (138)$$

$$\left[\lambda_{\theta 1}(s)\right]\hat{\sigma}_1(s) = \lambda_2(\underline{\alpha}_2(s)) \cdot\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\right],\ s\in[0,Y_1) \quad (139)$$

$$\left[\lambda_{\theta 1}(s)\right]\hat{\sigma}_1(s) \leq \lambda_2(\underline{\alpha}_2(s)) \cdot\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\right],\ s\in[Y_1,\infty) \quad (140)$$

The pair of (139) and (140) implies (136). If (71) is are satisfied, we have $\tau_1\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\theta_2(s) \leq \hat{\alpha}_1(s)$. This inequality together with the definition Y_1 implies $\lim_{s\to\infty}\underline{\alpha}_2^{-1}\circ\theta_2(s) \leq Y_1$. Thus, substitution of (139) into the left hand side of (138) results in (137). Hence, the proof is completed if λ_i, $i = 1, 2$ given in (81)-(82) solve (138), (139) and (140). Combining (81) with (82), we arrive at (139). Due to (82), (131) and the definition of $\lambda_{\theta 1}$, the property (86) leads to (140). On the other hand, from (71), (88) and (89) it follows that λ_1 in (81) solves (138). In the *(N2)* case, $\hat{\sigma}_1(\infty) < \infty$ follows from (75). The property (91) guarantees $\lambda_i(\infty) < \infty$, $i = 1, 2$, in (81) and (82). Define $\sigma_{cl,i} = \lambda_i(\infty)\sigma_{ri} \in \mathcal{P}_0$, and we do not need θri. *(i)*: The properties (76) and (77) allow us to define θ_2 as in (128) with $Y_2 = \infty$. If $\hat{\alpha}_1 \in \mathcal{K}_\infty$ holds, θ_1 can be defined as in (128) with $Y_1 = \infty$. When $\hat{\alpha}_1 \notin \mathcal{K}_\infty$ and *(M1)* hold, the properties (71) and (73) imply $\tau_1\circ\hat{\sigma}_1(\infty) = \hat{\sigma}_1(\infty) = \hat{\alpha}_1(\infty)$. Thus, θ_1 can be defined as in (128) with $Y_1 = \infty$. When $\hat{\alpha}_1 \notin \mathcal{K}_\infty$ and *(M2)* hold, the property (76) implies $\hat{\sigma}_1(\infty) > \tau_1^{-1}\circ\hat{\alpha}_1(\infty)$ yielding $\lambda_1(\infty) < \infty$. Although $\theta_1(s)$ defined by (128) is finite for $Y_1 < \infty$, the function $\lambda_{\theta 1}(s)$ defined by (129) is finite for all $s \in \mathbb{R}_+$.

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