A Lyapunov Approach to Cascade Interconnection of Integral Input-to-State Stable Systems

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***Abstract*—This paper deals with the stability of cascade interconnection of integral input-to-state (iISS) time-varying systems. A new technique to construct smooth Lyapunov functions of cascaded systems is proposed. From the construction, sufficient conditions for internal stability and stability with respect to external signals are derived. One of the derived conditions is a trade-off between slower convergence of the driving system and steeper input growth of the driven system. The trade-off is no more necessary if the speed of convergence of the driven system is not radially vanishing. The results are related to trajectory-based approaches and small-gain techniques for feedback interconnection. The difference between the feedback case and the cascade case is viewed from the requirement on convergence speed of autonomous parts.**

***Index Terms*—Cascaded systems, integral input-to-state stability, Lyapunov function. nonlinear systems.**

I. Introduction

Stability and stabilizability of cascaded nonlinear systems are often related to growth rate conditions on functions describing the interaction between systems (See [9], [14]–[19], [21], references and literature review therein). The cascade of input-to-state stable (ISS) systems is ISS since a growth rate condition can be always satisfied [23]. However, for broader classes of systems, stability of their cascade is not always guaranteed. Seibert and Suárez [19] derived global asymptotic stability (GAS) of a cascade of two time-invariant systems from individual GAS properties of the driving system and the disconnected driven system assuming that all solutions are bounded. To guarantee boundedness of the solutions, trade-off conditions between the decay rate of the driving system and the growth of the interconnection term in the driven system have been used in the literature. Although most of the trade-off conditions impose exponential decay rate on the driving system, the integrability condition on the driving signal can relax the exponential-decay constraint [16]. Roughly, the result in [16], [17] shows that integrability of the perturbing trajectory of the driving system is sufficient to ensure GAS of the cascade. This observation has been re-interpreted and re-written in [2] in terms of integral input-to-state stable stability (iISS) for a time-invariant cascade in which an iISS system is driven by a GAS system. The required trade-off condition is that the iISS gain of driven system needs to be steep satisfactorily in the direction toward the equilibrium if the convergence of the driving system is slow. Note that the set of iISS systems is larger and contains the ISS systems as a subset. The idea of the growth-order and decay-rate trade-off result has been improved further in [3] which shows additional conditions and states the trade-off in terms of Lyapunov-like inequalities (dissipation inequalities) of the two individual subsystems. These results dealing with time-invariant cascades of iISS and iISS/GAS systems are based on estimation of trajectories and give no explicit interpretation in terms of constructing Lyapunov functions of the cascades.

As for feedback interconnection, stability conditions for iISS systems have been derived in [5], [8]. The development is based on explicit construction of smooth Lyapunov functions of feedback systems. The relation between the feedback results [5], [8] and

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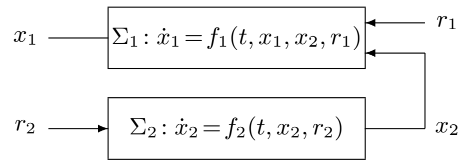

Fig. 1. Cascade system Σ.

the aforementioned cascade results has never been discussed in the literature yet. Constructing Lyapunov functions of cascades to fill the gap is the main aim of this paper. Developing constructive Lyapunov counterparts of [2], [3] by specializing the idea of [5], [8] in the cascade case, this paper covers time-varying systems which have not been straightforward in the previous iISS cascade approach. The authors of [2], [17] showed examples of cascades which are not GAS when their trade-off condition is not fulfilled. It is, however, also known that trade-off conditions are not always necessary. This paper demonstrates that the construction of Lyapunov functions can elucidate this fact in terms of non-vanishing convergence rate of the subsystems. In addition to addressing all these points, this paper corrects an error in [5].

In this paper, the set of positive definite functions from $\mathbb{R}_+ := [0,\infty)$ to $\mathbb{R}_+$, i.e., $\gamma(0) = 0$ and $\gamma(s) > 0$, $\forall s \in \mathbb{R}_+ \setminus \{0\}$, is denoted by $\mathcal{P}$. A function is said to belong to class $\mathcal{K}$ if it is in $\mathcal{P}$ and increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it tends to infinity as its argument approaches infinity. For $h \in \mathcal{P}$, we write $h \in \mathcal{O}(> L)$ with a non-negative real number L if there exists a positive real number $K > L$ such that $\limsup_{s\to 0+} h(s)/s^K < \infty$. We write $h \in \mathcal{O}(L)$ when $K = L$. Note that $\mathcal{O}(L) \subset \mathcal{O}(S)$ holds for $L > S$. The identity map on $\mathbb{R}$ is denoted by $\mathbf{Id}$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. A system is said to be GAS if it has a globally asymptotically stable equilibrium at the origin of the state space. UGAS stands for uniformly global asymptotic stability in the case of time-varying systems.

II. Motivating Examples

Consider the stability of the cascade

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{1}$$
$$\dot{x}_2 = -x_2^3 \tag{2}$$

motivated by similar examples extensively studied in the literature such as [16], [17]. The x_1-system is not ISS. Only iISS property holds. The x_2-system does not have the LES property which had been used constantly in the late 90s including Corollaries 2 and 3 in [2]. It is worth stressing that the LES constraint can be circumvented if the perturbing signal is integrable in the sense of [16], [17], which is not fulfilled by $x_2(t)$ of (2). If one considered $x_1 x_2 g_2(x_2)$ instead of $x_1 x_2$ in (1), the integrability of $|x_2(t) g_2(x_2(t))|$ from $t = 0$ to ∞ satisfied by an appropriate g_2 could guarantee the GAS of $x = [x_1, x_2]^T = 0$ of the modified system [16], [17]. Define $V_1(x_1) = \frac{1}{2}\log(x_1^2 + 1)$ and $V_2(x_2) = \frac{1}{2}x_2^2$. Then, the dissipation inequalities

$$\frac{\partial V_1}{\partial x_1} f_1 \le -\frac{x_1^2}{x_1^2+1} + |x_2|, \quad \frac{\partial V_2}{\partial x_2} f_2 \le -|x_2|^4 \tag{3}$$

are satisfied by (1) and (2). The input x_2 appears in the x_1-dissipation inequality in a first order fashion, which violates the coupled condition on convergence rate and gain growth required by Theorem 1 in [2]. This violation suggests that there exists an x_1-system such that its interconnection term is the same as (1) and that it generates unbounded $x_1(t)$ when (2) is connected. In fact, $\dot{x}_1 = -\mathrm{sgn}(x_1)\min\{1, |x_1|\} + x_1 x_2$ whose convergence term is saturated is such an example [2], [17]. It would be natural to expect some cascades to have GAS equilibriums without the help of the LES, integrability and growth rate assumptions if the x_1-system has non-saturated terms of the convergence. Alternatively, one would be able to search for a better bound of $\partial V_1/\partial x_1 \cdot f_1$ fulfilling such assumptions. Indeed, for the simple system (1)-(2), the choice $\partial V_1/\partial x_1 \cdot f_1 \leq -0.5x_1^2/(x_1^2+1)+|x_2|^4$ allows Theorem 1 in [3] to prove GAS of $x=0$. Although the approaches in [2], [3] do not aim at providing a Lyapunov function of the overall system, there should be a Lyapunov function we can construct explicitly. In fact,

$$V(x) = \frac{1}{4}\int_0^{V_1}\left(\frac{e^{2s}-1}{e^{2s}}\right)^4 ds + |x_2|^3 \tag{4}$$

is a Lyapunov function establishing the GAS property of (1)-(2). The idea of Corollary 1 *ii)* in this paper is to show that, for general systems including time-varying ones, such a Lyapunov function $V(x)$ can be directly constructed from the dissipation inequalities in (3) whenever the dissipation inequality of the x_1-system contains a radially non-vanishing convergent term. Constructing a Lyapunov function $V(x)$ is useful for robustness analysis with respect to external inputs. Consider $\dot{x}_1 = -x_1 + x_1x_2 + r_1^3$ and $\dot{x}_2 = -x_2^3 + r_2$. The time-derivative of $V(x)$ in (4) along the trajectories of the cascade is

$$\dot{V}(x) \leq -\frac{1}{20}\left(\frac{x_1^2}{x_1^2+1}\right)^5 - \frac{7}{4}|x_2|^5 + \frac{1}{8}|r_1|^3 + \frac{9}{5}|r_2|^{\frac{5}{3}} \ . \tag{5}$$

Therefore, the cascade system is iISS with respect to input (r_1, r_2) and state (x_1, x_2), and $V(x)$ is an iISS Lyapunov function. Theorem 2 in this paper demonstrates this point.

The next example of cascade is the following:

$$\dot{x}_1 = -\frac{x_1}{x_1^2+1} + x_2^2 + r_1, \quad x_1(0), x_2(0) \in \mathbb{R}_+ \tag{6}$$

$$\dot{x}_2 = -\frac{2x_2^4}{x_2^4+1} + \frac{r_2}{r_2+1}, \quad r_1(t), r_2(t) \in \mathbb{R}_+, \forall t \in \mathbb{R}_+. \tag{7}$$

The solutions $x(t) = [x_1(t), x_2(t)]^T$ evolve only in the positive orthant $\mathbb{R}_+^2$ for all $t \in \mathbb{R}_+$. For $r_1(t) = r_2(t) \equiv 0$, the system has a unique isolated equilibrium at $x=0$ on the boundary of $\mathbb{R}_+^2$. Its GAS property is defined by simply restricting the domain to $\mathbb{R}_+^2$. The x_1-system is not ISS, but only iISS with respect to input (x_2, r_1) and state x_1. The x_2-system is ISS with respect to input r_2 and state x_2. The convergence rate of x_2-system near the origin is much slower than LES. Since the system (6)-(7) is non-negative, the simplest choice of dissipation inequalities is

$$\frac{\partial V_1}{\partial x_1}f_1 \leq -\frac{x_1}{x_1^2+1} + x_2^2 + r_1, \tag{8}$$
$$\frac{\partial V_2}{\partial x_2}f_2 \leq -\frac{2x_2^4}{x_2^4+1} + \frac{r_2}{r_2+1}$$

for $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$. However, simple choices are not often useful in proving the stability of the cascade. Searching for suitable $V_1(x_1)$ and $V_2(x_2)$ is needed in the application of [2] and [3] for proving GAS and iISS of the cascade, respectively. What is a Lyapunov function characterizing the stability properties of the overall system? Under what condition is the construction possible? The idea of Theorem 1 in this paper is to provide answers in terms of the original $V_1(x_1)$ and $V_2(x_2)$. Theorem 1 with $\phi(s) = s^2$ yields an iISS Lyapunov function

$$V(x) = \frac{1}{5}\log(x_1^2+1) + \frac{1}{5}x_2^5 + x_2 \tag{9}$$

for (6)-(7) as might be expected. When the convergence rate in the dissipation inequality of the x_1-system decreases toward zero in the radial direction of x_1 as in (8), cascade connections are not always iISS. In this paper, a sufficient condition for the construction of an iISS Lyapunov function $V(x)$ will be derived and shown to be consistent with the result of [3].

III. Cascade of iISS Systems

Consider the nonlinear interconnected system Σ shown in Fig.1. The subsystems Σ_1 is driven by Σ_2. The state vector of Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The exogenous signals r_1 and r_2 are packed into $r = [r_1^T, r_2^T]^T \in \mathbb{R}^k$. This paper considers the following sets of systems.

Definition 1: Given $\alpha_1, \alpha_2 \in \mathcal{P}$, $\sigma_1 \in \mathcal{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathcal{K}\cup\{0\}$, we write $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$ if Σ_1 and Σ_2 are described by

$$\dot{x}_1 = f_1(t, x_1, x_2, r_1), \ x_i \in \mathbb{R}^{n_i}, \ r_i \in \mathbb{R}^{k_i} \tag{10}$$
$$\dot{x}_2 = f_2(t, x_2, r_2) \tag{11}$$
$$f_i(t, 0, \cdots, 0) = 0, \qquad t \in \mathbb{R}_+, \quad i = 1, 2 \tag{12}$$
$$f_i \text{ is locally Lipschitz in } (x, r_i) \text{ uniformly in } t$$
$$\text{and piecewise continuous in } t \tag{13}$$

which admit the existence of $\mathbf{C}^1$ functions V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2$, such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \bar{\alpha}_i(|x_i|), \quad i = 1, 2 \tag{14}$$
$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1 \leq -\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|) \tag{15}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}f_2 \leq -\alpha_2(|x_2|) + \sigma_{r2}(|r_2|) \tag{16}$$

hold for all $x \in \mathbb{R}^n$, $r \in \mathbb{R}^k$ and $t \in \mathbb{R}_+$.

The Lipschitzness imposed on f_i guarantees the existence of a unique maximal solution of Σ for locally essentially bounded r_i. The inequalities (15) and (16) are often referred to as dissipation inequalities, and their right hand sides are called supply rates. The individual system Σ_i fulfilling the above definition is said to be integral input-to-state stable (iISS) [22]. The function V_i is called a $\mathbf{C}^1$ iISS Lyapunov function [1]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is input-to-state stable (ISS) [20], and the function V_i is a $\mathbf{C}^1$ ISS Lyapunov function [24]. The trajectory-based definition of ISS (iISS) and the Lyapunov-based definition this paper adopts are equivalent in the sense of the existence of ISS (iISS, respectively) Lyapunov functions [22], [24]. By definition, an ISS system is always iISS. The iISS property guarantees GAS in the absence of the exogenous signal. The right hand side of (15) and (16) is independent of t, which means that this paper deals with iISS and ISS properties uniform in time [12]. ISS Lyapunov functions considered here is strict in the sense of [13]. In this paper, the convergence speed of the system Σ_i is said to be radially vanishing if $\liminf_{s\to\infty} \alpha_i(s) = 0$.

IV. Main Results

A. A Unified Characterization

This subsection considers the cascade Σ in the most general setting in this paper, and a constructive theorem for the stability of Σ is presented. The subsequent subsections derive specialized corollaries from the general result. To prepare for the explicit formula of a Lyapunov function for the cascade Σ given in Definition 1, we define $\hat{\sigma}_1$ as a class $\mathcal{K}$ function satisfying

$$\hat{\sigma}_1(s) = \sigma_1(s), \quad \forall s \in [0, 1) \tag{17}$$
$$\hat{\sigma}_1(s) \geq \sigma_1(s), \quad \forall s \in [1, \infty) \tag{18}$$
$$\sigma_1 \in \mathcal{K}_\infty \vee \alpha_1, \alpha_2 \in \mathcal{K}_\infty \ \Leftrightarrow \ \hat{\sigma}_1 \in \mathcal{K}_\infty \ . \tag{19}$$

Unless $\sigma_{r2}(s) \equiv 0$, let τ_2 be a class $\mathcal{K}_\infty$ function satisfying

$$\tau_2 \circ \sigma_{r2}(s) = (\omega_2 + 1)\sigma_{r2}(s), \quad \forall s \in [0, 1) \tag{20}$$

$$\tau_2 \circ \sigma_{r2}(s) > \sigma_{r2}(s), \quad \forall s \in [1, \infty) \tag{21}$$

for some real number $\omega_2 > 0$ and

$$\alpha_2 \in \mathcal{K}_\infty \vee \sigma_1 \notin \mathcal{K}_\infty \ \Rightarrow \ \tau_2(s) > s, \quad \forall s \in (0, \infty) \tag{22}$$

$$\alpha_2 \notin \mathcal{K}_\infty \wedge \sigma_1 \in \mathcal{K}_\infty \ \Rightarrow \ \lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \tau_2 \circ \sigma_{r2}(s). \tag{23}$$

We will not need τ_2 and ω_2 if $\sigma_{r2}(s) \equiv 0$. The existence of such a function τ_2 will be ensured by (27) $\vee$ (28) $\vee$ (29). For arbitrary $c_1 \in (0, 1/2)$, define

$$q(s) = \min_{w \in [\bar{\alpha}_1^{-1}(s), \underline{\alpha}_1^{-1}(s)]} c_1 \alpha_1(w) \ .$$

The following is the main theorem which constructs a Lyapunov function of the cascade Σ.

Theorem 1: Consider the cascaded systems Σ consisting of $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$ for given $\alpha_1 \in \mathcal{P}$, $\alpha_2, \sigma_1 \in \mathcal{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathcal{K} \cup \{0\}$ and positive integers n_1 and n_2. Assume that there exists $\phi \in \mathcal{K}_\infty$ such that

$$\frac{\phi(s)}{s} = 1 \ \vee \ \frac{\phi(s)}{s} \in \mathcal{K}_\infty \tag{24}$$

$$\int_1^\infty \frac{\phi \circ q(s)}{q(s)} ds = \infty \tag{25}$$

$$\lim_{s\to 0+} \frac{\phi \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} < \infty \tag{26}$$

are satisfied. Then, the following hold true.

i) If one of

$$\alpha_2 \in \mathcal{K}_\infty \tag{27}$$

$$\sigma_1 \notin \mathcal{K}_\infty \tag{28}$$

$$\infty > \lim_{s\to\infty} \alpha_2(s) \geq \sup_{s\in\mathbb{R}_+} \sigma_{r2}(s) \tag{29}$$

holds, the system Σ is iISS with respect to input r and state x. Furthermore, an iISS Lyapunov function V of Σ is

$$V(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s) ds + \int_0^{V_2(t,x_2)} \lambda_2(s) ds \tag{30}$$

$$\lambda_1(s) = \frac{c_1 \phi \circ q(s)}{q(s)}, \quad 0 < c_1 < \frac{1}{2} \tag{31}$$

$$\lambda_2(s) = \max_{w\in[0,s]} \frac{\phi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1}(w)}{(\tau_2 - \mathbf{Id}) \circ \tau_2^{-1} \circ \alpha_2 \circ \bar{\alpha}_2^{-1}(w)} \ . \tag{32}$$

Moreover, (26) and (32) can be replaced by

$$\int_0^1 \frac{\phi \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} ds < \infty \tag{33}$$

$$\lambda_2(s) = \begin{cases} \dfrac{\phi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1}(s)}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} & , s \in [0, 1) \\ \max\limits_{w\in[1,s]} \dfrac{\phi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1}(w)}{\alpha_2 \circ \bar{\alpha}_2^{-1}(w)} & , s \in [1, \infty) \end{cases} \tag{34}$$

respectively, if $\sigma_{r2}(s) \equiv 0$ or equivalently $r_2(t) \equiv 0$.

ii) If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ holds, the system Σ is ISS with respect to input r and state x. Furthermore, an ISS Lyapunov function of Σ is (30) given with (31) and (32).

Theorem 1 includes the ISS cascade addressed in [23] as the special case *ii)*. The construction of a Lyapunov function in [23] is less explicit and not completely specified. Theorem 1 can be considered as a result showing how to exploit the remaining flexibility for the ISS cascade to encompass the iISS cascade.

Remark 1: The assumption (24)$\wedge$(25)$\wedge$(26)$\wedge$$\{(27)\vee$(28)$\vee$(29)$\}$ is fulfilled if

$$\exists c_2 > 0, k \geq 1 \text{ s.t.}$$
$$c_2 \sigma_1 \circ \underline{\alpha}_2^{-1}(s) \leq [\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^k, \quad \forall s \in \mathbb{R}_+ \tag{35}$$

holds. In fact, the property $\sigma_1 \circ \underline{\alpha}_2^{-1} \in \mathcal{K}$ implies that (35) ensures $\lim_{s\to\infty} \sigma_1 \circ \underline{\alpha}_2^{-1}(s)/\alpha_2 \circ \bar{\alpha}_2^{-1}(s) < \infty$. Thus, the conditions (24), (25) and (26) hold with $\phi(s) = s$. Furthermore, the condition (35) always implies $\{$(27)$\vee$(28)$\vee$(29)$\}$. Hence, if (35) holds, the cascade system Σ is iISS with respect to input r and state x. The condition (35) is used by Corollary 3 i) in [5]. Theorem 1 relaxes (35). Note that the statements of ii)-v) of Corollary 3 in [5] are incorrect since their proofs are based on the non-vanishing assumption $\alpha_1, \alpha_2 \in \mathcal{K}$. Theorem 1 in this paper not only corrects the error, but also provides us with a more flexible Lyapunov function and a less restrictive proof specialized in the cascade system.

Remark 2: In the presence of the disturbance r_2, the assumption (27) $\vee$ (28) $\vee$ (29) cannot be removed for the choice of Lyapunov functions in the form of (30) whatever λ_1 and λ_2 are. To see this, suppose that (28) does not hold. Then, the property

$$\lambda_1(V_1(t,x_1))[-\alpha_1(|x_1|) + \sigma_1(|x_2|)]$$
$$-\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) \leq 0, \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \tag{36}$$

requires $\limsup_{s\to\infty} \alpha_2(s) = \infty \vee \limsup_{s\to\infty} \lambda_2(s) = \infty$. Next, suppose that $\limsup_{s\to\infty} \alpha_2(s) < \sup_{s\in\mathbb{R}_+} \sigma_{r2}(s)$ holds. Then, the existence of $\sigma_U \in \mathcal{P}_0$ and $c \in \mathbb{R}_+$ satisfying

$$\lambda_1(V_1(t,x_1))[-\alpha_1(|x_1|) + \sigma_1(|x_2|)]$$
$$+\lambda_2(V_2(t,x_2))[-\alpha_2(|x_2|) + \sigma_{r2}(|r_2|)]$$
$$\leq c + \sigma_U(|r_2|), \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, r_2 \in \mathbb{R}^{k_2} \tag{37}$$

implies $\limsup_{s\to\infty} \lambda_2(s) < \infty$. Therefore, when neither (28) nor (29) holds. the function $V(t,x)$ of the form (30) cannot be an iISS Lyapunov function unless (27) is satisfied.

Remark 3: Note that Theorem 1 does not allow $\liminf_{s\to\infty} \alpha_2(s) = 0$. As far as $\liminf_{s\to\infty} \alpha_2(s) > 0$ is concerned, we can assume $\alpha_2 \in \mathcal{K}$ without loss of generality since there always exists a class $\mathcal{K}$ function bounding from below. It is worth mentioning that, in the presence of the disturbance r_2, the assumption $\liminf_{s\to\infty} \alpha_2(s) > 0$ is necessary for the choice of Lyapunov functions in the form of (30) whatever λ_1 and λ_2 are. To see this, suppose that $\liminf_{s\to\infty} \alpha_2(s) = 0$. Since σ_1 is of class $\mathcal{K}$, the property (36) requires $\limsup_{s\to\infty} \lambda_2(s) = \infty$. In addition, due to $0 < \sup_{s\in\mathbb{R}_+} \sigma_{r2}(s)$, the existence of $\sigma_U \in \mathcal{P}_0$ and $c \in \mathbb{R}_+$ satisfying (37) implies $\limsup_{s\to\infty} \lambda_2(s) < \infty$ which contradicts the above consequence of (36). Thus, no function V in the form of (30) is an iISS Lyapunov function when $\liminf_{s\to\infty} \alpha_2(s) = 0$ holds.

Remark 4: Theorem 1 generalizes a similar result in [3] by covering time-varying systems and explicitly providing Lyapunov function of the whole system. It is stressed that the conditions (24) and (25) are fulfilled by $\phi(s) = s$. The remaining condition (26) for $\phi(s) = s$ is the growth order restriction used in [3]. The growth order restriction (26) involves the $\mathcal{K}_\infty$ bounds on V_i's, while the $\mathcal{K}_\infty$ bounds are not involved in a result of [3]. It is a natural consequence of constructing a Lyapunov function of the whole system. It is worth noting that this paper does not show iISS Lyapunov functions in the case of $\alpha_2(\infty) = 0$ although a Lyapunov functions for UGAS is derived in Corollary 1 presented later on. It is remarkable that the iISS property is proved in [3] without constructing a Lyapunov function of the cascade in the case of $\alpha_2(\infty) = 0$. The question of how to construct an iISS Lyapunov function for $\alpha_2(\infty) = 0$ remains open. According to Remark 3, we need to search for a Lyapunov function in which V_1 and V_2 are coupled.

B. Radially Non-Vanishing Case

The subsection deals with Σ_1 and Σ_2 whose convergence speed is not radially vanishing. The following demonstrates that the trade-off condition (24) $\wedge$ (25) $\wedge$ (26) is not necessary when $\alpha_1, \alpha_2 \in \mathcal{K}$. In other words, there always exists $\phi \in \mathcal{K}_\infty$ such that (24)$\wedge$(25)$\wedge$(26) holds.

Theorem 2: Consider the cascaded systems Σ consisting of $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$ for given $\alpha_1, \alpha_2, \sigma_1 \in \mathcal{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathcal{K} \cup \{0\}$ and positive integers n_1 and n_2. If one of (27), (28) and (29) is satisfied, the system Σ is iISS with respect to input r and state x. Furthermore, an iISS Lyapunov function of Σ is (30) given with (31), (32), $\phi(s) = [\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \bar{\sigma}_1^{-1}(s)]s$ and any $\bar{\sigma}_1 \in \mathcal{K}_\infty$ satisfying

$$\bar{\sigma}_1(s) = \hat{\sigma}_1(s), \quad \forall s \in [0, 1) \tag{38}$$

$$\bar{\sigma}_1(s) \geq \hat{\sigma}_1(s), \quad \forall s \in [1, \infty) \ . \tag{39}$$

C. Null Exogenous Signal Case

If the exogenous signal r_1 is not involved, we can remove the constraint $\alpha_2 \in \mathcal{K}$ from Theorem 1. When neither r_1 nor r_2 is involved, the trade-off condition is removed using Theorem 2 if the convergence speed of the driven system is not radially vanishing.

Corollary 1: Consider the cascaded systems Σ consisting of $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$ for given $\alpha_1, \alpha_2 \in \mathcal{P}$, $\sigma_1 \in \mathcal{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathcal{K} \cup \{0\}$ and positive integers n_1 and n_2. Then, the following hold true.

i) If there exists $\varepsilon > 0$ and $\hat{\alpha}_2 \in \mathcal{K}$ such that

$$\hat{\alpha}_2(s) \leq \alpha_2(s), \quad s \in [0, \varepsilon) \tag{40}$$

$$\int_0^1 \frac{\sigma_1 \circ \underline{\alpha}_2^{-1}(s)}{\hat{\alpha}_2 \circ \bar{\alpha}_2^{-1}(s)} ds < \infty \tag{41}$$

hold, the system Σ with $r_2(t) \equiv 0$ is iISS with respect to input r_1 and state x. Furthermore, an iISS Lyapunov function of Σ is

$$V(t, x) = \int_0^{V_1(t,x_1)} \lambda_1(s) ds + \int_0^{\xi(V_2(t,x_2))} \lambda_2(s) ds \tag{42}$$

$$\lambda_2(s) = \begin{cases} \dfrac{\phi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \xi^{-1}(s)}{\hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \xi^{-1}(s)} & , s \in [0, 1) \\ \displaystyle\max_{w \in [1, s]} \frac{\phi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \xi^{-1}(w)}{\hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \xi^{-1}(w)} & , s \in [1, \infty) \end{cases} \tag{43}$$

$$\xi(s) = \int_0^s \max\left\{ \frac{\hat{\alpha}_2 \circ \underline{\alpha}_2^{-1}(w)}{\alpha_2 \circ \underline{\alpha}_2^{-1}(w)}, 1 \right\} dw \tag{44}$$

with (31) and $\phi(s) = s$.

ii) If $\alpha_1 \in \mathcal{K}$, the system Σ is UGAS for $r_i(t) \equiv 0$, $i = 1, 2$. Furthermore, a UGAS Lyapunov function of Σ is (42) with (31), (43)-(44), $\phi(s) = [\hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \bar{\sigma}_1^{-1}(s)]s$ and any $\bar{\sigma}_1 \in \mathcal{K}_\infty$, $\hat{\alpha}_2 \in \mathcal{K}$ satisfying (38), (39) and (40).

The condition (41) constrains the growth of interconnection term in the driven system to be slow enough to cope with a low speed convergence of the driving system near the equilibrium. This type of trade-off condition (41) conforms to the main result of [2] which imposes restriction on growth of the interconnection term to establish GAS of time-invariant Σ. The condition (41) is essentially the same as the corresponding condition in [3]. More precisely, the condition (41) is less restrictive than the latter when $\underline{\alpha}_2 \sim \overline{\alpha}_2$ although (41) become conservative if the gap between $\underline{\alpha}_2$ and $\overline{\alpha}_2$ around zero is large.

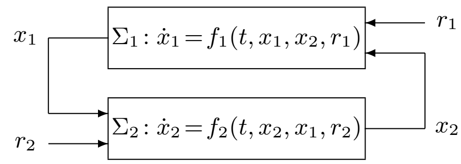

Fig. 2. Feedback system Σ_F.

V. Difference between Cascade and Feedback

Consider the feedback system shown in Fig.2, where $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_2, \sigma_{r2})$ defined with an additional term $+\sigma_2(|x_1|)$ in (16) for $\sigma_2 \in \mathcal{K}$. A special case of the feedback is cascade connection. Indeed, if we assume $\sigma_2(s) \equiv 0$ in Fig.2, the system is identical with Fig.1. This fact surely implies that stability of cascade systems can be verified by means of the small-gain technique. If Σ_1 and Σ_2 are individually ISS, the small-gain condition in [11] is met by zero loop-gain since the loop is broken. Thus, the cascade of ISS subsystems is always proved to be ISS. However, in the case of iISS subsystems, the application of the zero loop-gain to the small-gain result in [5], [8] does not yield a tight stability condition for the cascade. The following indicates this fact.

Theorem 3: Consider the feedback system Σ_F shown in Fig.2. Let n_1, n_2 be positive integers. Assume that $\alpha_1, \alpha_2 \in \mathcal{P}$, $\sigma_1, \sigma_2 \in \mathcal{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathcal{K} \cup \{0\}$ are $\mathbf{C}^1$ and satisfy

$$\alpha_i \in \mathcal{O}(>1), \quad \sigma_i, \sigma_{ri} \in \mathcal{O}(>0), \quad i = 1, 2. \tag{45}$$

Then, the system Σ_F with $r_i(t) \equiv 0$, $i = 1, 2$, is GAS for all $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i = 1, 2$, only if

$$\liminf_{s \to \infty} \alpha_i(s) > 0, \quad i = 1, 2 \ . \tag{46}$$

The above theorem demonstrates that, as long as we derive GAS, iISS and ISS from supply rates of the subsystems, the convergence rate of each subsystem needs to be radially non-vanishing ($\alpha_1, \alpha_2 \in \mathcal{K}$ without loss of generality) if the interconnection forms a closed loop. The smoothness of functions and (45) are only for proving the necessity in Theorem 3 among subsystems having unique maximal solutions. If systems are defined on the positive(or negative) orthant $\mathbb{R}_+^{n_i}$, the assumption $\alpha_i \in \mathcal{O}(>1)$ can be relaxed into $\alpha \in \mathcal{O}(1)$.

The necessity of $\alpha_1, \alpha_2 \in \mathcal{K}$ does not hold any more in the cascade case. There are pairs of supply rates from which we can derive stability of their cascade even if α_i are only positive definite. A simple example is the cascade connection of $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$ satisfying $\liminf_{s \to \infty} \alpha_1(s) = 0$ and $\sigma_1 = c\alpha_2$ for a constant $c > 0$ since $V(t, x) = V_1(t, x_1) + 2cV_2(t, x_2)$ is an iISS Lyapunov function. The results in Section IV cover this case. The proposed approach specialized in cascade connection is far beyond the application of small-gain technique in [5], [8] to cascaded systems since Theorem 1 and Corollary 1 allow the driven system to have radially vanishing convergence rate. In the case of $r_2(t) \equiv 0$, the driving system is also allowed to have radially vanishing convergence rate.

VI. Concluding Remarks

This paper has investigated stability of cascade interconnection of subsystems which are not necessary ISS. It has been shown how to construct iISS/UGAS Lyapunov functions of the cascades, thereby developing constructive Lyapunov counterparts of [2], [3]. This paper has demonstrated that a smooth Lyapunov function can be constructed explicitly if the condition of the trade-off between convergence rate and input growth rate holds, which is consistent with [2], [3]. The role of radially non-vanishing convergence rate of the driven system removing the trade-off has also been elucidated. Furthermore, this paper has addressed the difference between feedback interconnection and cascade interconnection. In distinction from the small-gain techniques, the proposed Lyapunov function specialized in cascade interconnection can allow the convergence speed of individual subsystems to be radially vanishing. Finally, it is mentioned that this paper does not show iISS Lyapunov functions in the case of $\alpha_2(\infty)=0$ in contrast to the GAS Lyapunov function in Corollary 1. Recently, the iISS property in the case of $\alpha_2(\infty)=0$ has been proved by [3] without driving a Lyapunov function of the cascade. Construction of an iISS Lyapunov function explicitly in the case of $\alpha_2(\infty)=0$ is a subject of future study. This is also an important step toward constructing Lyapunov functions iteratively for cascades constituted of more than two iISS systems as general as the non-constructive method [3]. In addition, for ISS systems, the relation between the approach in this paper and the construction of locally Lipschitz continuous Lyapunov functions, e.g., [4], [10] is worth investigating.

Appendix A
Proof of Theorem 1

i) The properties (24), (25), (26) (17) and (20) guarantee that there exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \bar{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{47}$$

holds for V defined by (30), (31) and (32). In the case of $\alpha_2 \notin \mathcal{K}_\infty \wedge \sigma_1 \notin \mathcal{K}_\infty$, the properties $\lim_{s\to\infty}\alpha_2(s) > 0$, (19), (20) and (32) imply $\sup_{s\in\mathbb{R}_+}\lambda_2(s) < \infty$. Since the properties $\tau_2 \in \mathcal{K}_\infty$ and (22) yield $s/(\tau_2 - \mathbf{Id})\circ\tau_2^{-1}(s) \geq 1$ for all $s \in \mathbb{R}_+$, there exists $h_{r2} > 0$ satisfying

$$\begin{aligned}\lambda_2(V_2)\{-\alpha_2(|x_2|) + \sigma_{r2}(|r_2|)\} \\ \leq -\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2) + h_{r2}\sigma_{r2}(|r_2|)\end{aligned} \tag{48}$$

in the case of $\alpha_2 \notin \mathcal{K}_\infty \wedge \sigma_1 \notin \mathcal{K}_\infty$. Next, suppose $\alpha_2 \in \mathcal{K}_\infty$. Define $\lambda_{\theta r2}(s) = \lambda_2\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_2\sigma_{r2}(s)$. From (32) and its non-decreasing property, we obtain

$$\begin{aligned}&\lambda_2(V_2)\{-\alpha_2(|x_2|) + \sigma_{r2}(|r_2|)\} \\ &\leq \begin{cases} -\lambda_2(V_2)\{\alpha_2(|x_2|) - \tau_2^{-1}\circ\alpha_2(|x_2|)\} \\ \qquad\qquad\qquad \text{if } \alpha_2(|x_2|) \geq \tau_2\sigma_{r2}(|r_2|) \\ -\lambda_2(V_2)\alpha_2(|x_2|) + \lambda_{\theta r2}(|r_2|)\sigma_{r2}(|r_2|) \quad \text{otherwise}\end{cases} \\ &\leq -[(\tau_2 - \mathbf{Id})\circ\tau_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}(V_2)]\lambda_2(V_2) \\ &\qquad\qquad + \lambda_{\theta r2}(|r_2|)\sigma_{r2}(|r_2|) \\ &\leq -\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2) + \lambda_{\theta r2}(|r_2|)\sigma_{r2}(|r_2|)\ .\end{aligned} \tag{49}$$

In the case of $\alpha_2 \notin \mathcal{K}_\infty \wedge \sigma_1 \in \mathcal{K}_\infty$, the property (23) ensures that $\lambda_{\theta r2}(s)$ is well-defined for all $s \in \mathbb{R}_+$, and we arrive at (49) again. Consider the case of $\phi(s)/s = 1$. We obtain

$$\begin{aligned}\lambda_1(V_1)\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\} \\ \leq -c_1\alpha_1(|x_1|) + c_1\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2) + c_1\sigma_{r1}(|r_1|)\end{aligned} \tag{50}$$

from (31) and $\hat{\sigma}_1 \in \mathcal{K}$. In the case of $\phi(s)/s \in \mathcal{K}_\infty$, define $Y \in \mathcal{K}_\infty$ by $Y^{-1}(s) = c_1\phi(s)/s$. Then, using $Y^{-1}\circ q(s) = \lambda_1(s)$, (31) and $\hat{\sigma}_1 \in \mathcal{K}$, we can verify $Y(\lambda_1(s))\lambda_1(s) = c_1\phi\circ q(s)$ and $Y^{-1}(\hat{\sigma}_1(|x_2|))\hat{\sigma}_1(|x_2|) = c_1\phi\circ\hat{\sigma}_1(|x_2|) \leq c_1\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2)$. Recall that $ab \leq Y(a)a + Y^{-1}(b)b$ holds for $a, b \in \mathbb{R}_+$ and $Y \in \mathcal{K}_\infty$. The definition (31) yields

$$\begin{aligned}\lambda_1(V_1)\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\} \\ \leq -(1-2c_1)\phi\circ q(V_1(x_1)) + c_1\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2(x_2)) \\ + c_1\,\phi\circ\sigma_{r1}(|r_1|)\ .\end{aligned} \tag{51}$$

Using (48), (49), (50), (51) and $0 < c_1 < 1/2$, we arrive at

$$\begin{aligned}\lambda_1(V_1)\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\} \\ + \lambda_2(V_2)\{-\alpha_2(|x_2|) + \sigma_{r2}(|r_2|)\} \\ \leq -\alpha_{o,1}(V_1) - \eta_2\phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(V_2) \\ + \sigma_{o,1}(|r_1|) + \sigma_{o,2}(|r_2|)\end{aligned}$$

for some $\alpha_{o,1} \in \mathcal{P}$, $\sigma_{o,1}, \sigma_{o,2} \in \mathcal{K}\cup\{0\}$ and $\eta_2 > 0$. Hence, V in (30) is an iISS Lyapunov function of Σ with respect to input r and state x. If $\sigma_{r2}(s) \equiv 0$ or $r_2(t) \equiv 0$, τ_2 and $\lambda_{\theta r2}$ vanish. Since it implies that the above arguments do not require λ_2 to be non-decreasing, the iISS can be established by (34). Note that (33) and (34) ensure $\lim_{V_2\to 0}\lambda_2(V_2)\alpha_2(\bar{\alpha}_2^{-1}(V_2)) = 0$ and (47).
ii) By virtue of $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, (19) and $\phi \in \mathcal{K}_\infty$, we have $\phi\circ q, \phi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1} \in \mathcal{K}_\infty$.

Appendix B
Proof of Theorem 2

From $\alpha_2 \in \mathcal{K}$ and $\bar{\sigma}_1 \in \mathcal{K}_\infty$ it follows that $\phi(s) \in \mathcal{K}_\infty$ and $\phi(s)/s \in \mathcal{K}_\infty$. Due to $\bar{\sigma}_1^{-1}\circ\sigma_1(s) \leq s$, we have $\phi\circ\sigma_1\circ\underline{\alpha}_2^{-1}(s)/\alpha_2\circ\bar{\alpha}_2^{-1}(s) \leq \sigma_1\circ\underline{\alpha}_2^{-1}(s)$. Since $\alpha_1 \in \mathcal{K}$ implies $q = c_1\alpha_1\circ\bar{\alpha}_1^{-1} \in \mathcal{K}$, we have $(\phi\circ q)/q \in \mathcal{K}$. Thus, (24) $\wedge$ (25) $\wedge$ (26) holds. Hence, Theorem 1 completes the proof.

Appendix C
Proof of Corollary 1

Define $\hat{V}_2 = \xi(V_2)$. Then, $\xi\circ\underline{\alpha}_2(|x_2|) \leq \hat{V}_2(t,x_2) \leq \xi\circ\bar{\alpha}_2(|x_2|)$ and $\xi \in \mathcal{K}_\infty$ hold. From (44) it follows that $d\hat{V}_2/dt \leq -\hat{\alpha}_2(|x_2|)$. Substituting $\hat{\alpha}_2$, $\xi\circ\underline{\alpha}_2$ and $\xi\circ\bar{\alpha}_2$ for α_2, $\underline{\alpha}_2$ and $\bar{\alpha}_2$, respectively. we obtain Claim *i)* from Theorem 1 *i)* with $\phi(s) = s$. Here, (41) implies (33). Note that (27) $\vee$ (29) is met for $r_2(t) \equiv 0$, equivalently $\sigma_{r2}(s) \equiv 0$. Theorem 2 proves Claim *ii)*.

Appendix D
Proof of Theorem 3

Consider Σ_F with $r_i(t) \equiv 0$, $i = 1, 2$. Suppose that $\liminf_{s\to\infty}\alpha_i(s) = 0$ holds for at least one of $i = 1, 2$. Due to $\sigma_1, \sigma_2 \in \mathcal{K}$, there exist $l_i > 0$ and $\delta_i > 0$ for $i = 1, 2$ such that

$$|x_i| = l_i,\ |x_{3-i}| \geq l_{3-i} \Rightarrow (1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|)$$

holds. Using [6, Lemma 1], choose a pair $f_1(x_1, u_1, r_1)$, $f_2(x_2, u_2, r_2)$: $\mathbb{R}^{n_i}\times\mathbb{R}^{m_i}\times\mathbb{R}^{k_i} \to \mathbb{R}$ for which there exist $\mathbf{C}^1$ functions V_1, V_2: $\mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_1$, $\bar{\alpha}_1$, $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $\underline{\alpha}_i(|x_i|) = V_i(x_i) = \bar{\alpha}_i(|x_i|)$ and

$$(1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|)\ \Rightarrow\ \frac{\partial V_i}{\partial x_i}f_i > \delta_i\alpha_i(|x_i|)$$

hold with $r_i(t) \equiv 0$ for $i = 1, 2$. These systems Σ_i, $i = 1, 2$, defined with $\dot{x}_i = f_i$ satisfy

$$|x_i| = l_i,\ |x_{3-i}| \geq l_{3-i} \Rightarrow \frac{\partial V_i}{\partial x_i}f_i > \delta_i\alpha_i(|x_i|)\ . \tag{52}$$

The pair of (52), $i = 1, 2$, implies that trajectories starting from $(x_1(0), x_2(0)) \in \{(x_1, x_2) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : V_i(x_i) \geq V_i(l_i), i = 1, 2\}$ stay there for all $t \in \mathbb{R}_+$. This implies that Σ_F is not GAS.

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