Robust Stability of Networks of iISS Systems: Construction of Sum-Type Lyapunov Functions

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***Abstract*—This paper gives a solution to the problem of verifying stability of networks consisting of integral input-to-state stable (iISS) subsystems. The iISS small-gain theorem developed recently has been restricted to interconnections of two subsystems. For large-scale systems, stability criteria relying only on gain-type information that were previously developed address only input-to-state stable (ISS) subsystems. To address the stability problem involving iISS subsystems interconnected in general structure, this paper shows how to construct Lyapunov functions of the network by means of a sum of nonlinearly rescaled individual Lyapunov functions of subsystems under an appropriate small-gain condition.**

***Index Terms*—Large-scale systems, Integral input-to-state stability, Nonlinear systems, Lyapunov functions, Robust stability criteria.**

I. Introduction

The notion of input-to-state stability (ISS) introduces the concept of nonlinear gain between input and state in order to deal with systems which do not admit finite linear gain [1]. This notion is useful in stability and robustness analysis of large-scale systems since system components are often incompatible with linear-like properties. Decomposition of a system into subsystems allowing for infinite linear gain sometimes reduces conservativeness arising in stability and robustness analysis [2], [3]. However, requiring bounded state for arbitrary magnitude of input is still restrictive. For instance, modules of biological networks, communication networks, network computing, air traffic models and neural network models are often not ISS. The notion of integral input-to-state stability (iISS) is a way to remove the limitation of ISS [4], and considering networks of iISS subsystems broadens the horizon of stability theory. In contrast to ISS, the notion of iISS allows us to cope with saturation mechanisms which often arise in practical networks (see e.g., [5]–[12]). Trajectories of mere iISS systems do not remain finite for large magnitude inputs, i.e., the ISS nonlinear gain cannot be defined. Difficulties of dealing with non-ISS systems have pushed forward the development of new theoretical tools [8], [13]–[16].

In contrast to networks of ISS systems for which a number of small-gain-type results have become available recently, e.g. [17]–[23], only a few attempts have been made for networks of iISS systems. Most Lyapunov-based studies on ISS small-gain criteria have employed the max-type construction for networks whose Lyapunov function V is defined as the weighted maximum of Lyapunov functions of individual subsystems V_i:

$$V(x) = \max_{i\in\{1,2,\dots,n\}} W_i(V_i(x_i)), \tag{1}$$

where n is the number of subsystems in a network. This function (1) was first employed for interconnected ISS systems with $n = 2$ in [24]. The weights are represented by the nonlinear functions W_i. In contrast, there have been only a few results on the construction of sum-type Lyapunov functions for networks whose Lyapunov function is defined as the nonlinearly-weighted sum of Lyapunov functions of individual subsystems:

$$V(x) = \sum_{i=1}^{n} W_i(V_i(x_i)) \tag{2}$$

A problem of constructing a function of the form (2) was posed for general networks consisting of ISS subsystems in [23] although no solution was derived. Recently, it has been proved in [25] that the max-type construction (1) does not yield a Lyapunov function if the network contains subsystems prescribed only by pure iISS properties which are not ISS[1]. The sum-type construction (2) has some clear advantages over the max-type construction since it yields smooth Lyapunov functions directly and it is applicable to networks involving iISS subsystems which are not ISS. Historically, both constructions belong to the basic idea of deriving scalar Lyapunov functions from vector Lyapunov functions [26]–[28]. Significant contributions have been made by using linear W_is mostly for linear systems. The class of nonlinear networks for which the sum-type construction is solved has been limited to trivial cases exhibiting explicit energy-type conservation or finite linear gain systems such as finite L_p gain systems (see , e.g., [23], [29]). It was found recently that the (nonlinearly-weighted) sum-type construction could give Lyapunov functions explicitly for feedback and cascade connection of two iISS subsystems [14], [30], [31]. In the presence of more than two subsystems, the technique proposed there could be

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[1] If there exists a max-type Lyapunov function guaranteeing the stability of all networks prescribed by given iISS dissipation inequalities of subsystems, then all the subsystems are already ISS.

extended to only a specific structure of networks [32], i.e., cactus graphs.

An attempt to tackle iISS networks was made in [33] and their investigation agrees that new tools are needed when the network involves non-ISS subsystems. The problem of guaranteeing stability of such a network remains unsolved [25]. In [8], as a useful idea of circumventing the difficulty of tackling the direct iISS formulation, a time embedded formulation aiming at verifying input-to-output stability is introduced in a trajectory-based setup. The ISS small-gain condition can still be used for non-ISS subsystems by assuming that the behavior of the subsystems is ISS after a transient period and that a trajectory estimate of the network during the period is available in a desired manner.

The purpose of this paper is to present a small-gain criterion for networks consisting of iISS subsystems interconnected in general graph structure. To the best of our knowledge, a methodology leading to the construction of ISS or iISS sum-type Lyapunov functions for general networks is presented for the first time. Section II is dedicated to preliminaries. The stability analysis of general iISS networks is formulated and recast as a problem of constructing a sum-type Lyapunov function in Section III. A solution is presented in Section IV. Section V illustrates its mechanism through networks of a simple structure, and provides technical keys to the main result. Section VI gives insights into the methodology by taking another formulation. Examples are shown in Section VII. Section VIII contains the conclusion. Proofs are given in the appendix. Reading the proof of Theorem 3 prior to that of the main theorem 2 would be comfortable.

II. Preliminaries

A. Notation and Convention

The symbol $|x|$ denotes the Euclidean norm of a real vector $x\in\mathbb{R}^n$. A continuous function $\omega : \mathbb{R}_+ := [0,\infty)\to\mathbb{R}_+$ is said to be positive definite if it satisfies $\omega(0)=0$ and $\omega(s)>0$ holds for all $s>0$. For a positive definite function $\omega : \mathbb{R}_+\to\mathbb{R}_+$, we write $\omega \in \mathcal{J}$ if it is non-decreasing. A function $\omega \in \mathcal{J}$ is said to be of class $\mathcal{K}$ (written as $\omega \in \mathcal{K}$) if it is strictly increasing; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. We write $\omega \in \mathcal{K} \cup \{0\}$ to indicate that ω is either of class $\mathcal{K}$ or the zero function. The symbol $\mathbf{Id}$ denotes the identity function on $\mathbb{R}_+$. Composition of $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is written as $\gamma_1 \circ \gamma_2$. For brevity, we adopt a nonstandard symbol for repeated composition as $\bigodot_{i=1}^{n} \gamma_i = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Let e_k for $k = 1, 2, ..., n$ be the standard basis of $\mathbb{R}^n$. Let I be an index set such that $I \subset \{1, 2, ..., n\}$. We denote by $P_I : \mathbb{R}^n \to \mathbb{R}^{\#I}$ the projection of the coordinates in $\mathbb{R}^n$ corresponding to the indices in I onto $R^{\#I}$, where $\#I$ is the cardinality of I. The anti-projection corresponding to P_I is $Q_I : \mathbb{R}^{\#I} \to \mathbb{R}^n$ defined as $x \in \mathbb{R}^{\#I} \mapsto (x_1 e_{i_1} + \ldots + x_{\#I} e_{i_{\#I}}) \in \mathbb{R}^n$, where $x = [x_1, ..., x_{\#I}]^T$ and $I = \{i_1, ..., i_{\#I}\}$. For a mapping $M : \mathbb{R}^n \to \mathbb{R}^n$, we use the similar notation $M_{I,J} := P_I \circ M \circ Q_J$. For a vector $\mathbf{s} \in \mathbb{R}^n$, we write $\mathbf{s}_I := P_I(\mathbf{s})$. For vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The negation of $a \geq b$ is denoted by $a \ngeq b$, i.e., there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. The relation $a \gg b$ is defined by $a_i > b_i$ for all $i = 1, \ldots, n$. The negation $a \not\gg b$ is the existence of an $i \in \{1, \ldots, n\}$ for which $a_i \leq b_i$ holds. Let $\overline{\mathbb{R}}_+$ denote the set of extended non-negative real numbers, i.e., $\overline{\mathbb{R}}_+ := [0, \infty]$. The mapping $\overline{M}_{I,J}$ is defined for $\overline{M} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$ as done on $\mathbb{R}_+^n$ above. The inequalities $<$ and $\leq$ on $\mathbb{R}_+$ are extended to $\overline{\mathbb{R}}_+$ with the condition $\infty \leq \infty$.

If ω is a class $\mathcal{K}_\infty$ function, the inverse ω^{-1} is of class $\mathcal{K}_\infty$. For $\omega \in \mathcal{K} \setminus \mathcal{K}_\infty$, its inverse ω^{-1} is defined on the finite interval $[0, \lim_{\tau\to\infty} \omega(\tau))$ since the continuous function ω is strictly increasing and $\omega(0) = 0$. For $\omega \in \mathcal{K}$, an operator $\omega^{\ominus}$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as

$$\omega^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \geq \omega(v)\}.$$

That is, we have $\omega^{\ominus}(s) = \infty$ for $s \geq \lim_{\tau\to\infty} \omega(\tau)$, and $\omega^{\ominus}(s) = \omega^{-1}(s)$ elsewhere. For $\omega \in \mathcal{J}$, the extension ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as

$$\omega(s) := \sup_{v\in\{w\in\mathbb{R}_+ : w\leq s\}} \omega(v).$$

Using these conventions for ω, $\gamma \in \mathcal{K}$, we have $\omega \circ \gamma^{\ominus}(s) = \lim_{\tau\to\infty} \omega(\tau)$ for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$. The identity $\omega^{\ominus} = \omega^{-1} \in \mathcal{K}$ holds if and only if $\omega \in \mathcal{K}_\infty$. It is important that, in the case of $\omega \in \mathcal{K} \setminus \mathcal{K}_\infty$, we have only $\omega \circ \omega^{\ominus}(s) \leq s$ for $s \in \overline{\mathbb{R}}_+$ although $\omega^{\ominus} \circ \omega(s) = s$ for $s \in \overline{\mathbb{R}}_+$. The convention of the "extended" inverse allows us to avoid repeating long lists of conditions with limit expressions which are considered many times in this paper. The convention is also equipped with the following equivalence which allows us to get rid of writing many similar inequalities with functions replaced in a cyclic order [34].

Proposition 1: **(a)** For $\gamma, \omega \in \mathcal{K} \cup \{\zeta^{\ominus} : \zeta \in \mathcal{K}\}$, the following two properties are equivalent:

$$\gamma \circ \omega(s) \leq s, \quad \forall s \in \mathbb{R}_+, \tag{3}$$

$$\omega \circ \gamma(s) \leq s, \quad \forall s \in \mathbb{R}_+. \tag{4}$$

(b) For $\gamma \in \mathcal{K} \cup \{\zeta^{\ominus} : \zeta \in \mathcal{K}\}$ and $\omega \in \mathcal{J}$, the two properties (3) and (4) are equivalent.

B. Terminology of Graphs

We shall introduce basic concepts of directed graphs [38]. The term "directed" is omitted when it is clear from the context. In this paper, the vertex set and the arc set of a directed graph G are denoted by $\mathcal{V}(G)$ and $\mathcal{A}(G)$, respectively. A vertex is said to be isolated if it has no arcs that are either directed towards or away from it. A self-loop is an arc that connects a vertex to itself. A walk is a finite alternating sequence of vertices and connecting arcs, beginning and ending with a vertex. The length of a walk is the number of arcs in the sequence. A walk is said to be a closed walk if the sequence starts and ends at the same vertex. A path is a walk that has no repeated vertices. A cycle is a walk that starts and ends at the same vertex but otherwise has no repeated vertices. Given a walk u of length k, this paper employs the following notation:

$$|u| = k, \quad u = (u(1), u(2), ..., u(k), u(k+1)), \tag{5}$$

where $u(i)$s are the vertices of u. The starting vertex of u is $u(k+1)$, and the ending vertex is $u(1)$. The arcs defining u are indicated by the adjacent ordered pairs in the above ordered list. An arc directed away from vertex j and directed toward vertex i is denoted by the ordered pair (i, j). If u is a cycle or a closed walk of length k, we have $u(1) = u(k+1)$. The length of cycles and paths is larger than or equal to one. The length of a cycle is one if and only if the cycle consists of a self-loop. In this paper, sequences consisting of a single vertex are referred to as neither cycles nor paths. A graph is said to be strongly connected if for every pair of distinct vertices i and j there is a path from vertex i to vertex j and a path from vertex j to vertex i. A graph G is said to be a complete directed graph if each ordered pair of vertices is connected by an arc. A graph defined with a subset of $\mathcal{V}(G)$ and a subset of $\mathcal{A}(G)$ is called a subgraph of G. Note that a pair of chosen subsets of $\mathcal{V}(G)$ and $\mathcal{A}(G)$ defines a graph only if all vertices involved in the chosen subset of $\mathcal{A}(G)$ belong to the chosen subset of $\mathcal{V}(G)$. A subgraph of G is called a cycle graph (resp,. a path graph) in G if it is defined by the vertices and arcs of a single cycle (resp,. a single path) in G. A singleton graph is such that it consists of a single vertex with no arcs. By the length of a cycle graph (resp., a path graph) U, we mean the length of the cycle (resp., the path) u represented by (5), and we write $|U| = k$.

Let $\mathcal{C}(G)$ denote the set of all cycle graphs in G. Let $\mathcal{P}(G)$ denote the set of all path graphs in G. Without ambiguity, the symbol $\mathcal{V}(G)$ also denotes the set of all singleton graphs contained in G. We use the notation $\mathcal{CP}(G) = \mathcal{C}(G) \cup \mathcal{P}(G)$ and $\mathcal{CPV}(G) = \mathcal{C}(G) \cup \mathcal{P}(G) \cup \mathcal{V}(G)$ for brevity. Let $\mathcal{I}(G)$ denote the set of all isolated vertices in G, and the set of the corresponding singleton graphs. We write $|U| = 0$ for $U \in \mathcal{V}(G)$. The union $G_1 \cup G_2$ of two graphs G_1 and G_2 is defined by $\mathcal{V}(G_1 \cup G_2) = \mathcal{V}(G_1) \cup \mathcal{V}(G_2)$ and $\mathcal{A}(G_1 \cup G_2) = \mathcal{A}(G_1) \cup \mathcal{A}(G_2)$. By the definition of subgraphs, a graph can be always covered by a union of its subgraphs, i.e.,

$$G = \bigcup_{U \in \mathcal{H}} U \tag{6}$$

holds for an appropriate set $\mathcal{H}$ of subgraphs of G. In particular, using cycle graphs, path graphs and singleton graphs, we can choose $\mathcal{H} = \mathcal{CPV}(G)$. The covering set $\mathcal{H}$ achieving (6) is not unique as multiple subgraphs belonging to a set $\mathcal{H}$ can overlap each other.

III. Problem Formulation

A. Network of iISS Systems

Consider a network Σ consisting of n subsystems Σ_i, $i = 1, 2, ..., n$ where $n \geq 2$. Let $x = [x_1^T, \dots, x_n^T]^T \in \mathbb{R}^N$ be the state vector of Σ, where the state vector of each subsystem is $x_i \in \mathbb{R}^{N_i}$, and $N := \sum N_i$ holds. Suppose that the dynamics of the i-th subsystem Σ_i is governed by

$$\Sigma_i: \ \dot{x}_i = f_i(x_1, \dots, x_n, r), \tag{7}$$

where $r \in \mathbb{R}^M$ and $f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}$. For each $i \in \{1, 2, ..., n\}$, the subsystem (7) is assumed to have a unique maximal solution $x_i(t)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any locally L^∞-inputs $x_j : \mathbb{R}_+ \to \mathbb{R}^{N_j}, j \neq i$, and $r : [0, \infty) \to \mathbb{R}^M$. For instance, this can be guaranteed by local Lipschitz condition on f_i. Using $f = [f_1^T, \dots, f_n^T]^T : \mathbb{R}^{N+M} \to \mathbb{R}^N$, the overall network Σ is written as

$$\Sigma: \ \dot{x} = f(x, r). \tag{8}$$

The knowledge of f is not assumed. Instead, this paper assumes that a dissipation inequality of each subsystem Σ_i is known as follows:

Assumption 1: For each $i = 1, 2, ..., n$, there exist a $\mathbf{C}^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and continuous functions $\alpha_i \in \mathcal{K}$, $\sigma_{i,j}, \kappa_i \in \mathcal{K} \cup \{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|), \quad x_i \in \mathbb{R}^{N_i} \tag{9}$$

$$\dot{V}_i \leq -\alpha_i(|x_i|) + \sum_{j=1}^{n} \sigma_{i,j}(|x_j|) + \kappa_i(|r|) \tag{10}$$

hold along the trajectories $x_i(t)$ for all $x_j \in \mathbb{R}^{N_j}, j \neq i$ and all $r \in \mathbb{R}^M$, where $\sigma_{i,i} \equiv 0$, $i = 1, 2, ..., n$.

Inequality (10) is called a dissipation inequality and means that each subsystem Σ_i with the inputs x_j, $j \neq i$ and r is integral input-to-state stable (iISS), and that V_i is an iISS Lyapunov function for the disconnected Σ_i [35]. A subsystem Σ_i prescribed by (10) is guaranteed to be input-to-state stable (ISS) with the inputs $x_j, j \neq i$ and r if and only if Assumption 1 can be satisfied with $\alpha_i \in \mathcal{K}_\infty$ for that i [35]–[37][2]. Thus, if $\alpha_i \in \mathcal{K}_\infty$, V_i is guaranteed to be an ISS Lyapunov function. By definition [4], the set of ISS systems is a strict subset of the set of iISS systems. The goal is to construct an iISS Lyapunov function $V(x)$ of the network Σ with respect to input r and state x, and to find a condition under which such construction is possible.

Remark 1: The function V_i is qualified as an iISS Lyapunov function of Σ_i even when α_i is only positive definite [35]. This paper assumes $\alpha_i \in \mathcal{K}$ which is a strict subset of positive definite functions. It is proved in [31] that a feedback interconnection made of iISS subsystems defined with the dissipation inequalities (10) with $\sigma_{j,i} \in \mathcal{K}$ is guaranteed to be iISS only if the function α_i can be bounded from below by a class $\mathcal{K}$ function. For cascade interconnections, $\alpha_i \in \mathcal{K}$ is not necessary. Such relaxation is not covered by this paper.

Remark 2: A subsystem Σ_i prescribed by (10) is guaranteed to be ISS if and only if there exist β_i, $\chi_{i,j}$, $\chi_i \in \mathcal{K}$ ($\chi_{i,i} = 0$) and a $\mathbf{C}^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ satisfying the implication

$$|x_i| \geq \sum_{j=1}^{n} \chi_{i,j}(|x_j|) + \chi_i(|r|) \ \Rightarrow \ \dot{V}_i \leq -\beta_i(|x_i|). \tag{11}$$

The characterization (11) for subsystems referred to as the implication formulation is used for ISS networks in [17], [19]–[21] with some equivalent variations in the conditional $|x_i| \geq \sum_{j=1}^{n} \chi_{i,j}(|x_j|) + \chi_i(|r|)$. The formulation is also called the gain margin formulation in [20]. If subsystems are not ISS, they cannot be defined in such an implication formulation [35].

[2]The existence of V_i is considered here. If V_i is fixed, the requirement corresponding to $\alpha_i \in \mathcal{K}_\infty$ for Σ_i to be ISS is that $\lim_{s\to\infty} \alpha_i(s) = \infty \vee \lim_{s\to\infty} \alpha_i(s) \geq \lim_{s\to\infty} \sum_{j=1}^{n} \sigma_{i,j}(s) + \kappa_i(s)$.

B. Sum-Type Lyapunov Functions

Using the functions $\mathbb{R}_+ \to \mathbb{R}_+$ given in Assumption 1, we define A, S, D, Λ: $\mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$A(\mathbf{s}) = \begin{bmatrix} \alpha_1 \circ \overline{\alpha}_1^{-1}(s_1) \\ \alpha_2 \circ \overline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \alpha_n \circ \overline{\alpha}_n^{-1}(s_n) \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{bmatrix} \begin{pmatrix} \overline{\alpha}_1^{-1}(s_1) \\ \overline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \overline{\alpha}_n^{-1}(s_n) \end{pmatrix}$$

$$S(\mathbf{s}) = \begin{bmatrix} \sum_j \sigma_{1,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \sum_j \sigma_{2,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \vdots \\ \sum_j \sigma_{n,j} \circ \underline{\alpha}_j^{-1}(s_j) \end{bmatrix} = \begin{bmatrix} 0 & \sigma_{12} & \cdots & \sigma_{1,n} \\ \sigma_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_{n-1,n} \\ \sigma_{n,1} & \cdots & \sigma_{n,n-1} & 0 \end{bmatrix} \begin{pmatrix} \underline{\alpha}_1^{-1}(s_1) \\ \underline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \underline{\alpha}_n^{-1}(s_n) \end{pmatrix}$$

$$D(\mathbf{s}) = \begin{bmatrix} s_1 + \delta_1(s_1) \\ s_2 + \delta_2(s_2) \\ \vdots \\ s_n + \delta_n(s_n) \end{bmatrix}, \quad \Lambda(\mathbf{s}) = \begin{bmatrix} \lambda_1(s_1) \\ \lambda_2(s_2) \\ \vdots \\ \lambda_n(s_n) \end{bmatrix},$$

where $\mathbf{s} = [s_1, s_2, ..., s_n]^T \in \mathbb{R}_+^n$. Note that the right-hand side of the definitions of A and S are not matrix operations since the entries are functions. The matrix-like representation helps us see the structure. Indeed, A, D and Λ have the same diagonal structure while S is not diagonal. The functions λ_i and δ_i have yet to be determined. The following is a result in [25].

Theorem 1: Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions δ_i, $i = 1, 2, ..., n$, such that

$$\lambda_i(s) > 0, \quad \forall s \in (0, \infty), \qquad i = 1, 2, ..., n \tag{12}$$

$$\int_0^\infty \lambda_i(s) ds = \infty, \qquad i = 1, 2, ..., n \tag{13}$$

$$\{\lim_{s\to\infty} \alpha_i(s) = \infty \ \vee \ \limsup_{s\to\infty} \lambda_i(s) < \infty\}, \ i=1,2,...,n \tag{14}$$

$$\Lambda(\mathbf{s})^T [-D^{-1} \circ A(\mathbf{s}) + S(\mathbf{s})] \leq 0, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \tag{15}$$

hold. Then the network Σ is iISS with respect to input r and state x. If both

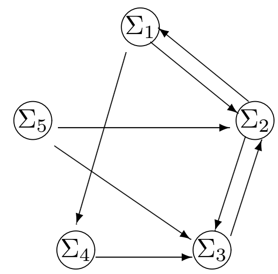
$$\alpha_i \in \mathcal{K}_\infty, \qquad i = 1, 2, ..., n \tag{16}$$

$$\liminf_{s\to\infty} \lambda_i(s) > 0, \quad i = 1, 2, ..., n \tag{17}$$

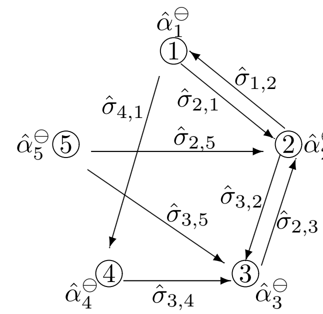
are satisfied additionally, then the network Σ is ISS. Furthermore, an iISS (ISS) Lyapunov function is

$$V(x) = \sum_{i=1}^n \int_0^{V_i(x_i)} \lambda_i(s) ds \ . \tag{18}$$

In this paper, the form (18) is referred to as the sum-type construction of Lyapunov functions. This sum form is the key to the success in constructing a Lyapunov function of the overall network involving non-ISS subsystems [25]. This paper constructs Λ explicitly, i.e. the functions W_i in (2) are computed explicitly in the next section.



(a) Network Σ.



(b) Weighted graph G.

Fig. 1. An example of a general network.

IV. General Networks: Main Result

A. A Solution

This section presents a main result which gives a solution $\Lambda(\mathbf{s})$ to the problem formulated in Theorem 1. In this paper, we let G denote the directed graph of the network Σ. The vertices of G are the subsystems Σ_i. The arcs are the signal flows between the subsystems. Thus, $\mathcal{V}(G) = \{1, 2, ..., n\}$. We have $(i,j) \in \mathcal{A}(G)$ if and only if $\sigma_{i,j} \neq 0$. The zero-nonzero "structure" of $S(\mathbf{s})$ corresponds to the adjacency matrix of G. The graph G has no self-loop. In order to associate the network Σ with a weighted directed graph, for any $U \in \mathcal{CPV}(G)$ let $J_U \in \mathbb{R}_+$ be a number such that

$$G = \bigcup_{W \in \{U \in \mathcal{CPV}(G):\ J_U \neq 0\}} W \quad . \tag{19}$$

Thus, the set of non-zero J_Us defines a covering $\mathcal{H} \subseteq \mathcal{CPV}(G)$ of the graph G in (6). The mappings J: $\mathcal{CPV}(G) \to \mathbb{R}_+$ represent not only the non-uniqueness of the covering $\mathcal{H}$, but also allow us to weight each subgraph in $\mathcal{H}$. We write J_U instead of $J(U)$ for brevity. For a set of $J_U \in \mathbb{R}_+$, $U \in \mathcal{CPV}(G)$, picked arbitrarily to fulfill (19), we can always compute $d_i > 0$ and $d_{i,j} > 0$ for $i \in \mathcal{V}(G)$ and $(i,j) \in \mathcal{A}(G)$ such that

$$1 = d_i \sum_{U \in \{W \in \mathcal{CPV}(G): \mathcal{V}(W) \ni i\}} J_U, \qquad \forall i \in \mathcal{V}(G) \tag{20}$$

$$1 = d_{i,j} \sum_{U \in \{W \in \mathcal{CP}(G): \mathcal{A}(W) \ni (i,j)\}} J_U, \qquad \forall (i,j) \in \mathcal{A}(G) \, . \tag{21}$$

Now, define $\hat{\alpha}_i \in \mathcal{K}$ and $\hat{\sigma}_{i,j} \in \mathcal{K} \setminus \{0\}$ for $i, j = 1, 2, ..., n$ as

$$\hat{\alpha}_i(s) = d_i \alpha_i(s), \quad \hat{\sigma}_{i,j}(s) = \begin{cases} d_{i,j}\sigma_{i,j}(s), & (i,j) \in \mathcal{A}(G) \\ 0 & , (i,j) \notin \mathcal{A}(G) \ . \end{cases} \tag{22}$$

The covering (19) decomposes the problem formulated in Theorem 1 into cycle graphs, path graphs and singleton graphs. However, in (15), the functions λ_i interlace multiple subgraphs in the set $\mathcal{H}$. The parameters J_U determines whether the sum is taken along a particular subgraph U in (15) and how large its contribution to (15) is. We define the weight of the arc (i,j) of G as the function $\hat{\sigma}_{i,j}$. We also define the weight of vertex i as the function $\hat{\alpha}_i^\ominus$. An example of a general network and its vertex-arc-weighted directed graph are illustrated in Fig.1. Due to the non-uniqueness of J_U in achieving (19), the weighted graph is not uniquely determined from the network

Σ. The solution in Theorem 1 depends on the choice of J_U through (22).

Using the next lemma, we build a weighted complete directed graph with self-loops from G by adding fictitious arcs and weights.

Lemma 1: Consider $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j} \in \mathcal{K} \cup \{0\}$, $\hat{\sigma}_{i,i} = 0$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i,j = 1,2,...,n$, satisfying

$$\left\{ \lim_{s\to\infty} \hat{\alpha}_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} \hat{\sigma}_{i,j}(s) < \infty \right\}, \quad j = 1,2,...,n. \tag{23}$$

Suppose that there exist $c_i > 1$, $i = 1,2,...,n$ such that

$$\bigodot_{i=1}^{|U|} \underline{\alpha}_{u(i)}^{-1} \circ \overline{\alpha}_{u(i)} \circ \hat{\alpha}_{u(i)}^{\ominus} \circ c_{u(i)} \hat{\sigma}_{u(i),u(i+1)}(s) \le s, \ \forall s \in \mathbb{R}_+ \tag{24}$$

holds for cycles of all $U \in \mathcal{C}(G)$. Let τ_i be such that

$$1 < \tau_i < c_i, \quad i = 1,2,...,n \tag{25}$$

is satisfied. Then there exist functions $F_{i,j} \in \mathcal{J}$, $i,j = 1,2,...,n$, satisfying

$$F_{i,j}(s) \ge \max\left\{ \max_{\substack{1 \le q \le n \\ q \ne i, q \ne j}} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q F_{q,j}(s), \ \hat{\sigma}_{i,j}(s) \right\}, \quad \forall s \in \mathbb{R}_+, \ i,j = 1,2,...,n \tag{26}$$

$$\underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ c_i F_{i,i}(s) \le s, \ \forall s \in \mathbb{R}_+, \ i = 1,2,...,n \tag{27}$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} \max\left\{ \max_{\substack{1 \le q \le n \\ q \ne i, q \ne j}} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q F_{q,j}(s), \ \hat{\sigma}_{i,j}(s) \right\} = \infty, \quad i,j = 1,2,...,n \tag{28}$$

$$\left\{ \lim_{s\to\infty} \hat{\alpha}_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} F_{i,j}(s) < \infty \right\}, \ j = 1,2,...,n. \tag{29}$$

Assumption (23) guarantees that the functions $F_{i,j}(s)$ do not attain ∞ for finite $s \in \mathbb{R}_+$. Given $F_{i,j} \in \mathcal{J}$ for all pairs (i,j), $i,j = 1,2,...,n$, we can define a weighted complete directed graph with self-loops. The weight functions $F_{i,j}$ are assigned to individual arcs connecting all possible pairs in $\mathcal{V}(G)$. Arcs that are not present in the original graph G will be called fictitious arcs. The function $F_{i,j}$ replaces the original weight $\hat{\sigma}_{i,j}$ if the arc (i,j) exists in the original graph G. The rearrangement of weights to define the complete graph is illustrated by Fig.2. This rearrangement allows us to obtain the functions λ_i achieving (15). The weights $F_{i,i}$ of self-loops are not used explicitly in the expression of λ_i, and only the existence of $F_{i,i} \in \mathcal{J}$ fulfilling (27) is essential.

Theorem 2: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$, $\sigma_{i,i} = 0$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i,j = 1,2,...,n$, satisfying

$$\left\{ \lim_{s\to\infty} \alpha_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} \sigma_{i,j}(s) < \infty \right\}, \quad j = 1,2,...,n. \tag{30}$$

Define $\hat{\alpha}_i \in \mathcal{K}$ and $\hat{\sigma}_{i,j} \in \mathcal{K} \cup \{0\}$, with $d_i, d_{i,j} > 0$ for $i,j = 1,2,...,n$ as in (20)-(22). Suppose that there exist $c_i > 1$, $i = 1,2,...,n$ such that (24) holds for cycles of all $U \in \mathcal{C}(G)$. Let τ_i and $\psi \ge 0$ be such that (25) and

$$\left(\frac{\tau_i}{c_i}\right)^{\psi} \le \tau_i - 1, \quad i = 1,2,...,n. \tag{31}$$

Pick $F_{i,j} \in \mathcal{J}$, $i,j = 1,2,...,n$, such that (26)-(29) are satisfied. Define $\overline{\lambda}_i \in \mathcal{J}$, $i = 1,2,...,n$, by

$$\overline{\lambda}_i(s) = \left[\frac{1}{\tau_i} \hat{\alpha}_i(\overline{\alpha}_i^{-1}(s))\right]^{\psi} \prod_{j \in \mathcal{V}(G) - \{i\}} \left[F_{j,i}(\underline{\alpha}_i^{-1}(s))\right]^{\psi+1}. \tag{32}$$

Let ν_i: $(0,\infty) \to \mathbb{R}_+$, $i = 1,2,...,n$, be continuous functions satisfying

$$0 < \nu_i(s) < \infty, \quad s \in (0,\infty) \tag{33}$$

$$\lim_{s\to\infty} \hat{\alpha}_i(s) = \infty \ \vee \ \lim_{s\to\infty} \nu_i(s) < \infty \tag{34}$$

$$\overline{\lambda}_i(s)\nu_i(s) : \text{non-decreasing and continuous for } s \in (0,\infty) \tag{35}$$

and

$$\nu_{u(j)} \circ \overline{\alpha}_{u(j)} \circ \hat{\alpha}_{u(j)}^{\ominus} \circ \tau_{u(j)} \hat{\sigma}_{u(j),u(j+1)}(s) \le \left(\frac{c_{u(j+1)}}{\tau_{u(j+1)}}\right)^{\psi} (\tau_{u(j+1)} - 1)\nu_{u(j+1)} \circ \underline{\alpha}_{u(j+1)}(s), \quad \forall s \in (0,\infty), \quad j = 1,2,...,|U| \tag{36}$$

for cycles of all $U \in \mathcal{C}(G)$. Then non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,...,n$, defined by

$$\lambda_i(s) = \overline{\lambda}_i(s)\nu_i(s), \quad s \in (0,\infty), \quad i = 1,2,...,n \tag{37}$$

$$\lambda_i(0) = \lim_{s\to 0^+} \overline{\lambda}_i(s)\nu_i(s) \tag{38}$$

achieve (12)-(15) and (17) with $\delta_i(s) = b_i s$, $i = 1,2,...,n$, for some $b_i > 0$.

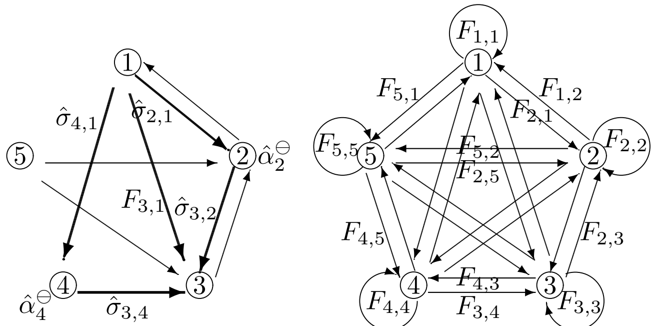
By virtue of the convention of $\hat{\alpha}_i^{\ominus}$ and Proposition 1, condition (24) is invariant under cyclic shifting of vertices. To obtain (24), the constants c_i cannot be arbitrarily large. It is stressed that, for any choice of constants $c_i > 1$ satisfying (24), the constants τ_i, ψ satisfying (25) and (31) and the functions ν_i, $i = 1,2,...,n$, satisfying (33)-(36) always exist and can be chosen easily (see Subsection IV-B for ν_i). The set of functions λ_i given in (37) yields an iISS Lyapunov function V in the form of (18). Theorem 2 demonstrates that the collection of the inequalities in (24) is a sufficient condition for iISS of the network Σ. The sufficient condition imposes the small-gain property (24) on all cycles in the "original" graph G.

It is verified easily that the small-gain condition (24) with $c_i > 1$ implies the existence of an integer $i \in \{1,2,...,|U|\}$ satisfying

$$\lim_{s\to\infty} \hat{\alpha}_{u(i)}(s) = \infty \ \vee \ \lim_{s\to\infty} \hat{\alpha}_{u(i)}(s) > \lim_{s\to\infty} \hat{\sigma}_{u(i),u(i+1)}(s) \tag{39}$$

for each cycle of $U \in \mathcal{C}(G)$. Due to (22), we obtain the existence of an $i \in \{1,2,...,n\}$ satisfying

$$\lim_{s\to\infty} \alpha_i(s) = \infty \ \vee \ \lim_{s\to\infty} \alpha_i(s) > \lim_{s\to\infty} \sum_{j=1}^{n} \sigma_{i,j}(s). \tag{40}$$



(a) Computing a weight $F_{3,1}$ of a fictitious arc. (b) Complete graph with $F_{i,j}$.

Fig. 2. Rearrangement of weights to obtain a fictitious complete graph.

Hence, condition (24) implies that at least one subsystem Σ_i should be ISS with respect to the combined input from the other subsystems. This fact conforms to a necessity condition derived in [34] for the stability of iISS networks. It is emphasized that a network can be iISS in the presence of multiple non-ISS systems [34] and such cases are encompassed by (24). For instance, examples in Section VII have two iISS subsystems which are not ISS and share the same cycle. Condition (30) can be proved to be necessary as long as the sum-type Lyapunov function (2) is employed as an iISS Lyapunov candidate for the entire network Σ unless we restrict the influence of disturbances r ,i.e., κ_i. Indeed, the following is proved:

Proposition 2: Given $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$ and $\kappa_i \in \mathcal{K}_\infty$, $i,j = 1,2,...,n$, suppose that there exists an iISS Lyapunov function with respect to input r and state x in the form of (2) for every Σ satisfying Assumption 1 [3]. Then property (30) holds.

B. Remarks

J_U *:* In view of (24), to achieve (19), there are better choices than $\{J_U \neq 0 \quad \Leftrightarrow \quad U \in \mathcal{CPV}(G)\}$ implying $\mathcal{H} = \mathcal{CPV}(G)$ in (6). Properties (20) and (21) yield[4] $1/d_i = J_i + \sum_{(i,j)\in\mathcal{A}(W)} 1/d_{i,j}$, where J_i is for the singleton graph of vertex i. Taking (24) into account, $J_i = 0$ is preferable unless $i \in \mathcal{I}(G)$. The value $J_i > 0$ does not influence (24) if vertex i is isolated. If two distinct cycle graphs U_1 and U_2 share a vertex, the choice $J_{U_1} \geq J_{U_2} > 0$ reduces constraint (24) for the cycle of U_1 relatively to U_2. The choice $\mathcal{H} = \{W \in \mathcal{P}(G) : |W| = 1\} \cup \mathcal{I}(G)$ gives a one-to-one correspondence between $J_U, U \in \mathcal{H} - \mathcal{I}(G)$ and $d_{i,j}$, $(i,j) \in \mathcal{A}(G)$. If one wants to reduce complexity, one may use $J_U = 1$ for all $U \in \mathcal{H} = \mathcal{C}(G) \cup \mathcal{Q}(G) \cup \mathcal{I}(G)$ defined with

$$\mathcal{Q}(G) := \left\{ W \in \mathcal{P}(G) : \begin{array}{l} \mathcal{A}(W) \cap \mathcal{A}(\tilde{W}) = \emptyset, \ \forall \tilde{W} \in \mathcal{C}(G) \\ \mathcal{A}(W) \not\subset \mathcal{A}(\hat{W}), \ \forall \hat{W} \in \mathcal{P}(G) - \{W\} \end{array} \right\},$$

and $J_U = 0$ otherwise. This is a way to reduce the number of non-zero J_Us.

$F_{i,j}$ *:* The functions $F_{i,j}$, $i \neq j$, can be computed by evaluating arcs with $\hat{\sigma}_{p,q}$s and vertices with $\hat{\alpha}_p^{\ominus}$s in all paths from j to i in G. We do not have to evaluate walks which are not paths. The functions $F_{i,i}$ can be computed by evaluating $\hat{\sigma}_{p,q}$s and $\hat{\alpha}_p^{\ominus}$s along all cycles starting and ending at i. In fact, the proof of Lemma 1 shows that (26)-(29) are satisfied by

$$F_{i,j}(s) = \max_{u\in\mathcal{CP}(i,j)} \hat{\sigma}_{u(1),u(2)} \circ \bigodot_{i=2}^{|u|} \underline{\alpha}_{u(i)}^{-1} \circ \overline{\alpha}_{u(i)} \circ \hat{\alpha}_{u(i)}^{\ominus} \circ \tau_{u(i)} \hat{\sigma}_{u(i),u(i+1)}(s), \quad i,j = 1,2,...,n, \tag{41}$$

where $\mathcal{CP}(i,j)$ denotes the set of all paths and cycles from vertex j to vertex i of G.

ν_i *:* Since we have (31), the simplest choice of continuous functions $\nu_i : (0,\infty) \to \mathbb{R}_+$, $i = 1,2,...,n$ satisfying (33)-(36) is $\nu_1(s) = \nu_2(s) = ... = \nu_n(s) = \text{constant} > 0$. A non-constant choice of ν_i, $i = 1,2,...,n$ is $\nu \circ F_{l,i} \circ \underline{\alpha}_i^{-1}(s)$ defined with any $l \in \mathcal{V}(G)$ and any non-decreasing continuous function ν satisfying $0 < \nu(s) < \infty$ for $s \in (0,\infty)$. Indeed, the properties (33)-(36) follow from $F_{l,i} \in \mathcal{J}$ and (29). The above examples of ν_is are non-decreasing functions. It is worth noting that decreasing functions are also eligible in contrast to the previous results [30], [32]. To see this point, consider the network Σ which is a cycle graph and satisfies $\gamma_j\hat{\alpha}_j(s) = \hat{\sigma}_{i,j}(s)$, $\gamma_j > 0$ and $\underline{\alpha}_i(s) = \overline{\alpha}_i(s) = s$ for $i = 1,2,...,n$. Then the choice

$$\nu_i(s) = g_i[\hat{\alpha}_i(s)]^{-n\psi-n+1}, \quad i = 1,2,...,n \tag{42}$$

fulfills (33)-(36) for appropriate constants $g_i > 0$. In this case, the functions λ_i, $i = 1,2,...,n$, become positive constants and the small-gain condition is $\gamma_1\gamma_2\cdots\gamma_n < 1$.

Cascades : If G contains no cycle, the problem (15) is always solvable. Since the small-gain condition (24) is required for cycles only, the functions $F_{i,j} \in \mathcal{J}$ are guaranteed to exist and the functions in (37) satisfy (12)-(15) and (17). It is stressed that this holds true under the assumption of $\alpha_i \in \mathcal{K}$, $i = 1,2,...,n$, and (30). This fact is consistent with the $n = 2$ result in [31].

Small-gain conditions : A condition similar to (24) has been developed for networks of ISS subsystems in [18]–[22]. If all subsystems are ISS, i.e., (16), then by virtue of Theorems 1 and 2, the function V in (18) with λ_i in (37) is an ISS Lyapunov function. Note that (30) is fulfilled by (16). It is stressed that the cyclic small-gain condition in [18]–[22] and the condition (24) in this paper are different from each other in general for $n > 2$ even if Σ_is are restricted to ISS subsystems. The cyclic small-gain condition is evaluated along all cycles in a decoupled manner, while condition (24) is evaluated

[3]To make sure that the functions f_i's are locally Lipschitz, it suffices to assume that α_i and σ_{ij} are continuously differentiable on $(0,\infty)$ and satisfy $\alpha_i \in \mathcal{O}(>1)$ and $\sigma_{ij} \in \mathcal{O}(>0)$ for $i,j = 1,2,...,n$, $j \neq i$. For $\omega \in \mathcal{K}\cup\{0\}$, we write $\omega \in \mathcal{O}(>L)$ with a non-negative number L if there exists a positive number $K > L$ such that $\limsup_{s\to 0+} \omega(s)/s^K < \infty$. See [30], [34].

[4]when vertex i is not the starting vertex of any path graph in $\mathcal{H}$. This footnote was missing on p.1197 of the TAC publication.

simultaneously for all the cycles. To put it another way, the cyclic small-gain condition [18]–[22] evaluates loop gains of cycles separately even if the cycles are overlapped. In contrast, condition (24) is given in terms of $\hat{\alpha}_i^{\ominus}$ (i.e., $\hat{\alpha}_i$) and $\hat{\sigma}_{i,j}$ which are only portions of α_i and $\sigma_{i,j}$ if the cycles are overlapped. The difference arises from variations in defining networks. See Section VI.

c_i *:* In (24), the constants $c_i - 1$ describe how much the loop gain of a cycle is smaller than the identity function in a linear way. At the expense of some technical complexity in the formula for λ_i, the small-gain condition (24) can be relaxed into the existence of $\omega_i \in \mathcal{K}_\infty$ satisfying

$$\bigodot_{i=1}^{|U|} \underline{\alpha}_{u(i)}^{-1} \circ \overline{\alpha}_{u(i)} \circ \hat{\alpha}_{u(i)}^{\ominus} \circ (\mathbf{Id} + \omega_{u(i)}) \circ \hat{\sigma}_{u(i),u(i+1)}(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{43}$$

for cycles of all $U \in \mathcal{C}(G)$. Property (24) just chooses the linear function $s + \omega_i(s) = c_i s$ in (43). All the results in this paper remain valid even if nonlinear gap functions $\omega_i \in \mathcal{K}_\infty$ are used instead of $(c_i - 1)s$. The formula for λ_i is omitted since the reader can refer to [30].

V. Cycle Networks: The Role of the Small-Gain Condition

The purpose of this section is twofold. One is to present a formula for the solution Λ to the stability problem (15) posed by Theorem 1 for cycle networks. The cycle network case is more intuitive than the case of general networks. The other is to illustrate the mechanisms to arrive at the solution to the general network problem presented in Section IV. This section gives a technique to embed a degree of freedom in Λ for cycle networks. The solution given in Theorem 2 consists of temporary solutions Λ to all cycles residing in the general network. The degree of freedom allows us to combine these temporary solutions to construct a single solution Λ to the general network problem.

This section assumes that Σ forms a cycle graph of length n, i.e., without loss of generality,

$$\hat{\alpha}_i = \alpha_i, \quad i = 1, 2, ..., n \tag{44}$$

$$\hat{\sigma}_{i,j} = \sigma_{i,j} \begin{cases} \neq 0 & j = (i \mod n) + 1 \\ = 0 & \text{otherwise} \end{cases} . \tag{45}$$

An example for $n = 5$ is shown in Fig.3(a). The next lemma generates weighting functions $F_{i,j}$ for all possible ordered pairs of vertices of the graph G associated with the cycle network Σ.

Lemma 2: Consider $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j} \in \mathcal{K}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, $j = (i \mod n) + 1$, satisfying

$$\{\lim_{s\to\infty} \hat{\alpha}_j(s) = \infty \ \vee \ \lim_{s\to\infty} \hat{\sigma}_{i,j}(s) < \infty\}, \quad j = 1, 2, ..., n, \ i = (j - 2 \mod n) + 1. \tag{46}$$

Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\bigodot_{i=1}^{n} \underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ c_i \hat{\sigma}_{i,i+1}(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{47}$$

holds with the notation $\hat{\sigma}_{n,n+1} = \hat{\sigma}_{n,1}$. Let τ_i be such that (25) is satisfied. Then there exist $F_{i,j} \in \mathcal{J}$, $i, j = 1, 2, ..., n$, such that

$$F_{i,j}(s) \geq \hat{\sigma}_{i,j}(s), \ \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n, \ j = (i \mod n) + 1 \tag{48}$$

$$F_{i,j}(s) \geq F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q \hat{\sigma}_{q,j}(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n, \ i + 2 \leq h \leq i + n, \quad q = (h - 2 \mod n) + 1, \quad j = (h - 1 \mod n) + 1 \tag{49}$$

$$\underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ c_i F_{i,i}(s) \leq s, \ \forall s \in \mathbb{R}_+, \ i = 1, 2, ..., n \tag{50}$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} \hat{\sigma}_{i,j}(s) = \infty, \quad i = 1, 2, ..., n, \ j = (i \mod n) + 1 \tag{51}$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q \hat{\sigma}_{q,j}(s) = \infty, \quad i = 1, 2, ..., n, \ i + 2 \leq h \leq i + n, \quad q = (h - 2 \mod n) + 1, \quad j = (h - 1 \mod n) + 1 \tag{52}$$

and (29) hold.

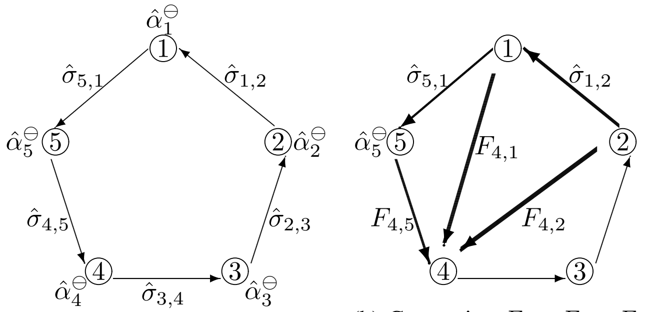
The process of computing $F_{i,j}$ is more intuitive than the general network case since (48) and (49) are constructive as illustrated in Fig.3(b). Each $F_{i,j}$ is computed by induction from $F_{i,i+1}$ then $F_{i,i+2}$ until it comes back to $F_{i,i}$. The small-gain condition (47) makes (50) achievable when the induction completes a cycle. The next theorem constructs λ_i with $F_{i,j}$, $i, j = 1, 2, ..., n$. The proposed formula for λ_is does not explicitly involve $F_{i,i}$. However, achieving (50) in terms of $F_{i,i}$ is the central mechanism for guaranteeing such a set of λ_is to be a solution of (15).

Theorem 3: Consider $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j} \in \mathcal{K}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, $j = (i \mod n) + 1$, satisfying (46). Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that (47) holds. Let τ_i and $\psi \geq 0$ be such that (25) and (31) are satisfied. Pick $F_{i,j} \in \mathcal{J}$, $i, j = 1, 2, ..., n$, as in Lemma 2. Define $\bar{\lambda}_i \in \mathcal{J}$, $i = 1, 2, ..., n$ by (32). Let ν_i: $(0, \infty) \to \mathbb{R}_+$, $i = 1, 2, ..., n$, be continuous functions satisfying (33)-(35) and

$$\nu_i \circ \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ \tau_i \hat{\sigma}_{i,j}(s) \leq \left(\frac{c_j}{\tau_j}\right)^{\psi} (\tau_j - 1) \nu_j \circ \underline{\alpha}_j(s), \quad \forall s \in (0, \infty), \quad j = (i \mod n) + 1. \tag{53}$$

Then the non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., n$, defined by (37) and (38) achieve (12)-(15) and (17) with (44), (45) and $\delta_i(s) = b_i s$, $i = 1, 2, ..., n$, for some $b_i > 0$.

Combining Theorem 3 with Theorem 1, the set of functions λ_i in (37) yields an iISS Lyapunov function V of the cycle network Σ in the form of (18). In this way, the small-gain condition (47) with the existence of $c_i > 1$, $i = 1, 2, ..., n$ plays a central role as a sufficient condition for the iISS of the cyclic interconnection. For cycle networks, it is not necessary to use the flexibility of λ_i provided by ν_i. Indeed, the choice $\nu_i(s) = 1$, $i = 1, 2..., n$ fulfills (33)-(35) and (53). When a network consists of overlapping cycles, the freedom of ν_i provided in Theorem 3 can be utilized to make multiple λ_is, that are computed independently with respect to individual cycles



(a) Weighted graph G.

(b) Computing $F_{4,5}$, $F_{4,1}$, $F_{4,2}$, ... up to $F_{4,4}$.

Fig. 3. Cycle network.

for the common single vertex i between those cycles, agree with each other. Then multiple properties (15) defined with respect to individual cycles are achieved simultaneously by requiring all cycles to satisfy small-gain conditions. Covering by subgraphs (19)-(21) ensures that combining individual (15) corresponding to the cycles leads to the original problem (15) for the general network. This is the mechanism of the solution given in Theorem 2 for general networks.

Remark 3: Inequality (47) generalizes the iISS small-gain condition developed for the two subsystems case [30], and the general condition (47) includes it as a special case. However, the formula for the λ_is given in Theorem 3 is different from the one given in [30]. The new solution provides the flexibility for adjusting the λ_is to cope with general structure of networks. In [14], [30], it is proved for feedback interconnection of two iISS subsystems that one subsystem is not required to be ISS, which means that $\hat{\alpha}_i^{-1}$ of that subsystem does not have to be defined on the whole of $\mathbb{R}_+$. Therefore, the small-gain condition using $\hat{\alpha}_i^{-1}$'s becomes asymmetric since one $\hat{\alpha}_i^{-1}$ is well-defined while the other $\hat{\alpha}_i^{-1}$ is not. The employment of $\hat{\alpha}_i^{\ominus}$'s allows us to write the small-gain condition in the symmetric way (47), even in the presence of mere iISS subsystems, which contrasts sharply with the expressions in [14], [30], [32].

Remark 4: The small-gain condition (47) requires that at least one subsystem in the cycle network is ISS with respect to the input from the adjacent subsystem. This requirement is justified by a necessity result in [34]. There can be many non-ISS subsystems in a cycle network as long as (47) holds. Condition (47) shows that the network is stable if the "weak" stability properties of non-ISS subsystems are compensated by the "strong" stability properties of at least one ISS subsystem. Section VII illustrates this fact by a cycle network containing two non-ISS subsystems.

VI. Another Formulation of Supply Rates

This section shows the following two points using a different formulation of supply rates for subsystems Σ_i: 1) The step (19)-(21) of covering by subgraphs is removed; 2) The small-gain stability criterion is equivalently expressed by a matrix-like condition generalizing an ISS result in [21]. We achieve these two points by replacing Assumption 1 with the following:

Assumption 2: For each $i = 1, 2, ..., n$, there exist a $\mathbf{C}^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and continuous functions $\alpha_i \in \mathcal{K}$, $\sigma_{i,j}, \kappa_i \in \mathcal{K} \cup \{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that (9) and

$$\dot{V}_i \leq -\alpha_i(|x_i|) + \max\left\{ \max_{j\in\{1,2,...,n\}} \sigma_{i,j}(|x_j|),\, \kappa_i(|r|) \right\} \quad (54)$$

hold along the trajectories $x_i(t)$ for all $x_j \in \mathbb{R}^{N_j}, j \neq i$ and all $r \in \mathbb{R}^M$, where $\sigma_{i,i} \equiv 0$, $i = 1, 2, ..., n$.

The above assumption employs the dissipation inequality (54) instead of (10) for each subsystem. The formulation of type (54) in the supply rates of subsystems Σ_i is called maximization aggregation in [21], while the formulation (10) is summation aggregation. We can verify that Theorem 1 holds true for (54) by redefining the operator $S : \mathbb{R}_+^n \to \mathbb{R}_+^n$ as

$$S(\mathbf{s}) = \begin{bmatrix} \max\limits_j \sigma_{1,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \max\limits_j \sigma_{2,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \vdots \\ \max\limits_j \sigma_{n,j} \circ \underline{\alpha}_j^{-1}(s_j) \end{bmatrix}, \quad \mathbf{s} \in \mathbb{R}_+^n. \quad (55)$$

The κ_i terms are absorbed by the operator D as done to obtain (15) from (10) in [25] with the help of $\max\{\max_j \sigma_{i,j},\, \kappa_i\} \leq \max_j \sigma_{i,j} + \kappa_i$. The graph G associated with the network Σ is defined with $\mathcal{V}(G)$ and $\mathcal{A}(G)$. The vertices $\mathcal{V}(G)$ are subsystems, $i = 1, 2, ..., n$. The pair (i, j) is an element of the arc set $\mathcal{A}(G)$ if and only if $\sigma_{i,j} \neq 0$, i.e., not identically zero. In the maximization formulation (54), we define the weight of the arc (i, j) of G as the function $\sigma_{i,j}(s)$. Thus, the functions $\hat{\sigma}_{i,j}$ in Fig.1 (b) are replaced by $\sigma_{i,j}$s. We have the following.

Lemma 3: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$, $\sigma_{i,i} = 0$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i, j = 1, 2, ..., n$, satisfying (30). Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\bigodot_{i=1}^{|U|} \underline{\alpha}_{u(i)}^{-1} \circ \overline{\alpha}_{u(i)} \circ \alpha_{u(i)}^{\ominus} \circ c_{u(i)} \sigma_{u(i),u(i+1)}(s) \leq s, \ \forall s \in \mathbb{R}_+ \quad (56)$$

holds for cycles of all $U \in \mathcal{C}(G)$. Let τ_i be such that (25) is satisfied. Then there exist $F_{i,j} \in \mathcal{J}$, $i, j = 1, 2, ..., n$, satisfying

$$F_{i,j}(s) \geq \max\left\{ \max_{\substack{1 \leq q \leq n \\ q \neq i, q \neq j}} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \alpha_q^{\ominus} \circ \tau_q F_{q,j}(s),\; \sigma_{i,j}(s) \right\}, \ \forall s \in \mathbb{R}_+, \ i, j = 1, 2, ..., n \quad (57)$$

$$\underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \alpha_i^{\ominus} \circ c_i F_{i,i}(s) \leq s, \ \forall s \in \mathbb{R}_+, \ i = 1, 2, ..., n \quad (58)$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} \max\left\{ \max_{\substack{1 \leq q \leq n \\ q \neq i, q \neq j}} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \alpha_q^{\ominus} \circ \tau_q F_{q,j}(s),\; \sigma_{i,j}(s) \right\} = \infty, \quad i, j = 1, 2, ..., n \quad (59)$$

$$\{\lim_{s\to\infty} \alpha_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} F_{i,j}(s) < \infty\}, \ j = 1, 2, ..., n. \quad (60)$$

The following is the main result in the maximization formulation (54) of subsystems.

Theorem 4: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$, $\sigma_{i,i} = 0$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i, j = 1, 2, ..., n$, satisfying (30). Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that (56) holds for cycles of all $U\in\mathcal{C}(G)$. Let τ_i and $\psi\geq 0$ be such that (25) and (31) are satisfied. Pick $F_{i,j}\in\mathcal{J}$, $i,j=1,2,...,n$, such that (57)-(59) and (60) are satisfied. Define $\overline{\lambda}_i\in\mathcal{J}$, $i=1,2,...,n$, by

$$\overline{\lambda}_i(s)=\left[\frac{1}{\tau_i}\alpha_i(\overline{\alpha}_i^{-1}(s))\right]^{\psi}\prod_{j\in\mathcal{V}(G)-\{i\}}\left[F_{j,i}(\underline{\alpha}_i^{-1}(s))\right]^{\psi+1}. \quad (61)$$

Let ν_i: $(0,\infty)\to\mathbb{R}_+$, $i=1,2,...,n$, be continuous functions satisfying (33)-(35) and

$$\nu_{u(j)}\circ\overline{\alpha}_{u(j)}\circ\alpha_{u(j)}^{\ominus}\circ\tau_{u(j)}\sigma_{u(j),u(j+1)}(s)\leq \left(\frac{c_{u(j+1)}}{\tau_{u(j+1)}}\right)^{\psi}(\tau_{u(j+1)}-1)\nu_{u(j+1)}\circ\underline{\alpha}_{u(j+1)}(s) \quad \forall s\in(0,\infty),\quad j=1,2,...,|U| \quad (62)$$

for cycles of all $U\in\mathcal{C}(G)$. Then the non-decreasing continuous functions $\lambda_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2,...,n$, defined by (37) and (38) achieve (12)-(15) and (17) with $\delta_i(s)=b_is$, $i=1,2,...,n$, for some $b_i>0$.

The above theorem demonstrates that V of the form (18) is an iISS Lyapunov function of the network Σ with input r and state x if λ_is are constructed as in (61). Collective condition (56) is sufficient for iISS of the network Σ. Compared with Theorem 2 for the summation of supply rates, Theorem 4 directly uses α_i and $\sigma_{i,j}$ appearing in the supply rates of subsystems. In other words, neither the stability criterion (56) nor the construction of the Lyapunov function V with (61) requires the process of covering the network graph by subgraphs, i.e., the computation of $\hat{\alpha}_i$ and $\hat{\sigma}_{i,j}$ in (20)-(22). Theorem 4 allows some subsystems to be non-ISS. It realizes the intuitive idea of compensating vulnerable subsystems with constraining subsystems in feedback for the general network topology. We are able to replace the linear functions c_is in (56) with nonlinear functions $s+\omega_i(s)$ as in Subsection IV-B. For the maximization formulation for supply rates of subsystems, the small-gain condition (56) can be shown to be equivalent to a matrix-like condition. This equivalence was demonstrated for ISS subsystems and $\mathcal{K}_\infty$ gain functions on $\mathbb{R}_+$ in [18], [22], [40]. In the iISS formulation this paper employs, the network is allowed to have multiple non-ISS subsystems which lead to the small-gain condition (56) containing several $\alpha_i^{\ominus}$ which are of neither $\mathcal{K}_\infty$ nor $\mathcal{K}$ and involve $\overline{\mathbb{R}}_+$. Define $N:\overline{\mathbb{R}}_+^n\to\overline{\mathbb{R}}_+^n$ by

$$N(\mathbf{s})=\begin{bmatrix}\overline{\alpha}_1\circ\alpha_1^{\ominus}(s_1)\\ \overline{\alpha}_2\circ\alpha_2^{\ominus}(s_2)\\ \vdots\\ \overline{\alpha}_n\circ\alpha_n^{\ominus}(s_n)\end{bmatrix}.$$

In the setting of the previous ISS results [22], [40] which amount to $\alpha_i\in\mathcal{K}_\infty$, $i=1,2,...,n$, in Assumption 2, we have $A\circ N(\mathbf{s})=\mathbf{s}$ for all $\mathbf{s}\in\mathbb{R}_+^n$. It is, however, stressed that admitting $\alpha_i\in\mathcal{K}\setminus\mathcal{K}_\infty$ implies $A\circ N\neq\mathbf{Id}$ on $\overline{\mathbb{R}}_+$ although $N\circ A=\mathbf{Id}$ holds on $\overline{\mathbb{R}}_+$. In spite of this fact, we have the following equivalence:

Lemma 4: Let $\alpha_i\in\mathcal{K}$, $\sigma_{i,j}\in\mathcal{K}\cup\{0\}$, $\underline{\alpha}_i,\overline{\alpha}_i\in\mathcal{K}_\infty$ and $\mathbf{Id}+\delta_i\in\mathcal{K}_\infty$ for $i,j=1,2,...,n$. Then the following three properties are equivalent to one another:

$$D\circ S(\mathbf{s})\not\geq A(\mathbf{s}),\quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\} \quad (63)$$

$$N\circ D\circ S(\mathbf{s})\not\geq \mathbf{s},\quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\} \quad (64)$$

$$S(\mathbf{s})\not\geq D^{-1}\circ A(\mathbf{s}),\quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\}. \quad (65)$$

Based on Lemma 4, we can verify that the small-gain condition (56) is equivalent to the matrix-like condition (63) with $s+\delta_i(s)=c_is$. This fact is stated for general $\delta_i\in\mathcal{K}_\infty$ as follows:

Proposition 3: Consider $\alpha_i\in\mathcal{K}$, $\sigma_{i,j}\in\mathcal{K}\cup\{0\}$, $\sigma_{i,i}=0$, $\underline{\alpha}_i,\overline{\alpha}_i\in\mathcal{K}_\infty$ and $\delta_i\in\mathcal{K}_\infty$ for $i,j=1,2,...,n$. Then the inequality

$$\bigodot_{i=1}^{|U|}\underline{\alpha}_{u(i)}^{-1}\circ\overline{\alpha}_{u(i)}\circ\alpha_{u(i)}^{\ominus}\circ(\mathbf{Id}+\delta_{u(i)})\circ\sigma_{u(i),u(i+1)}(s)<s,\quad \forall s\in\mathbb{R}_+\setminus\{0\} \quad (66)$$

holds for cycles of all $U\in\mathcal{C}(G)$ if and only if (64) is satisfied.

It is worth noting that there exist $\delta_i\in\mathcal{K}_\infty$, $i=1,2,...,n$, such that (66) holds for cycles of all $U\in\mathcal{C}(G)$ if and only if there exist (possibly different) $\delta_i\in\mathcal{K}_\infty$, $i=1,2,...,n$, such that

$$\bigodot_{i=1}^{|U|}\underline{\alpha}_{u(i)}^{-1}\circ\overline{\alpha}_{u(i)}\circ\alpha_{u(i)}^{\ominus}\circ(\mathbf{Id}+\delta_{u(i)})\circ\sigma_{u(i),u(i+1)}(s)\leq s,\quad \forall s\in\mathbb{R}_+ \quad (67)$$

is satisfied for cycles of all $U\in\mathcal{C}(G)$. Indeed, the strict inequality $<$ implies $\leq$ by definition. The converse can be proved by replacing $\delta_i(s)$ with $\delta_i(s)/2$.

The inequalities (63)-(65) can be also posed with $\not\gg$ instead of $\not\geq$. To be more precise, under the assumption that the graph G is strongly connected, there exist $\delta_i\in\mathcal{K}_\infty$, $i=1,2,...,n$, such that (63) holds if and only if there exist (possibly different) $\delta_i\in\mathcal{K}_\infty$, $i=1,2,...,n$, such that $D\circ S(\mathbf{s})\not\gg A(\mathbf{s})$ holds for all $\mathbf{s}\in\mathbb{R}_+^n$.

Remark 5: If all subsystems Σ_i are ISS, the existence of β_i, $\chi_{i,j}$, $\chi_i\in\mathcal{K}$ ($\chi_{i,i}=0$) and a $\mathbf{C}^1$ function $V_i:\mathbb{R}^{N_i}\to\mathbb{R}_+$ satisfying the implication

$$|x_i|\geq\max\left\{\max_j\chi_{i,j}(|x_j|),\,\chi_i(|r|)\right\}\ \Rightarrow\ \dot{V}_i\leq-\beta_i(|x_i|), \quad (68)$$

is an alternative to Assumption 2 [35]–[37]. The implication (68) is the maximization aggregation used in [21]. The implication form (68) is not valid if a subsystem Σ_i is not ISS. The dissipation form (54) is universally applicable to both ISS and non-ISS subsystems. For networks consisting of ISS subsystems described with the maximization aggregation, the ISS of the networks is guaranteed by the satisfaction of the cyclic small-gain condition developed in [18]–[21]. The cyclic small-gain condition is equivalent to the matrix-like condition proposed in [21] for networks consisting of ISS subsystems. Notice that the equivalence was proved only for the maximization aggregation [21]. The results of this paper allow us to generalize the equivalence to networks involving non-ISS subsystems. When $\alpha_i\in\mathcal{K}_\infty$ holds for all $i=1,2,...,n$, i.e., all subsystem are ISS, Theorem 1 for

Assumption 2 ensures that the function V constructed as in (18) with (61) is an ISS Lyapunov function of the network Σ with input r and state x. In the ISS case, the stability condition (56) reduces into the cyclic small-gain condition developed for ISS subsystems in [18]–[21]. Condition (63) is presented in [21] for ISS networks. It is stressed that the stability condition (24) for the summation supply rates is not precisely the same as the cyclic small-gain condition (56) (i.e., the one in [18]–[21]) even if all subsystem are ISS. See Subsection IV-B.

VII. EXAMPLES

Example 1: Consider the network Σ consisting of four subsystems defined with

$$\alpha_1(s)=\frac{6s}{s+1},\ \alpha_2(s)=4s,\ \alpha_3(s)=\frac{2s}{s+1},\ \alpha_4(s)=4s$$
$$\sigma_{1,2}(s)=2s,\ \sigma_{2,3}(s)=\frac{s}{2s+1},\ \sigma_{3,4}(s)=s$$
$$\sigma_{4,1}(s)=\frac{s}{s+1},\ \sigma_{2,1}(s)=\frac{s}{s+1}$$
$$\sigma_{1,1}(s)=\sigma_{1,3}(s)=\sigma_{1,4}(s)=\sigma_{2,2}(s)=\sigma_{2,4}(s)=0$$
$$\sigma_{3,1}(s)=\sigma_{3,2}(s)=\sigma_{3,3}(s)=\sigma_{4,2}(s)=\sigma_{4,3}(s)=\sigma_{4,4}(s)=0$$
$$\kappa_1(s)=\kappa_2(s)=\kappa_3(s)=\kappa_4(s)=s$$

and $\underline{\alpha}_i(s)=\overline{\alpha}_i(s)=s$, $i=1,2,...,4$, in Assumption 1. The subsystems Σ_1 and Σ_3 are not ISS, but iISS. We obtain $\hat{\alpha}_1(s)=3s/(s+1)$, $\hat{\alpha}_2(s)=2s$ and $\hat{\sigma}_{1,2}(s)=s$ by choosing J_U as

$$J_U=\begin{cases}1 & ,U\in\{(1,2,3,4,1),(1,2,1)\}\\ 0 & ,\ \text{otherwise}\ .\end{cases}$$

Here, cycles graphs are indicated by cycles. The remaining $\hat{\alpha}_i$ and $\hat{\sigma}_{i,j}$ are identical with α_i and $\sigma_{i,j}$, respectively. Using $c=c_1=c_2=c_3=c_4$, the small-gain condition (24) with respect to the cycles $(1,2,3,4,1)$ and $(1,2,1)$ is computed as

$$\frac{c^4s}{(48+6c^2-c^4)s+48}\le s,\quad \frac{c^2s}{(6-c^2)s+6}\le s,$$

respectively. These two conditions are satisfied by $1<c<\sqrt{6}$. Together with $\tau=\tau_1=\tau_2=\tau_3=\tau_4=9/4<c$, the prerequisites of Lemma 1 are satisfied, and we obtain

$$F_{2,1}(s)=\frac{s}{s+1},\ F_{3,1}(s)=\frac{\tau s}{4(s+1)},\ F_{4,1}(s)=\frac{s}{s+1}$$

and the other functions $F_{i,j}\in\mathcal{J}$ in the same manner by (41). Since with $\psi=0$ condition (31) is met, we obtain

$$\lambda_1(s)=\frac{\tau s^3}{4(s+1)^3},\ \lambda_2(s)=\frac{\tau^3s^3}{36}$$
$$\lambda_3(s)=\frac{\tau^3s^3}{12(2s+1)^3},\ \lambda_4(s)=\frac{\tau^3s^3}{2(\tau s+2)^2}$$

from (32), (37) and (38) with $\nu_1=\nu_2=\nu_3=\nu_4=1$. By virtue of Theorems 1 and 2, the above set of λ_is yields an iISS Lyapunov function of Σ as (18). Hence, the network Σ is iISS.

Example 2: Consider the cycle network consisting of four subsystems defined with

$$\alpha_1(s)=\frac{3s}{s+1},\ \alpha_2(s)=2s,\ \alpha_3(s)=\frac{2s}{s+1},\ \alpha_4(s)=4s$$
$$\sigma_{1,2}(s)=s,\ \sigma_{2,3}(s)=\frac{s}{s+1},\ \sigma_{3,4}(s)=s,\ \sigma_{4,1}(s)=\frac{s}{s+1},$$
$$\sigma_{1,1}(s)=\sigma_{1,3}(s)=\sigma_{1,4}(s)=\sigma_{2,1}(s)=\sigma_{2,2}(s)=\sigma_{2,4}(s)=0$$
$$\sigma_{3,1}(s)=\sigma_{3,2}(s)=\sigma_{3,3}(s)=\sigma_{4,2}(s)=\sigma_{4,3}(s)=\sigma_{4,4}(s)=0$$
$$\kappa_1(s)=\kappa_2(s)=\kappa_3(s)=\kappa_4(s)=s$$

and $\underline{\alpha}_i(s)=\overline{\alpha}_i(s)=s$, $i=1,2,...,4$, in Assumption 1. The subsystems Σ_1 and Σ_3 are not ISS, but iISS. In contrast to the previous example, we have $\hat{\alpha}_i=\alpha_i$ and $\hat{\sigma}_{i,j}=\sigma_{i,j}$ for all i,j since the network consists of a single cycle $(1,2,3,4,1)$. The small-gain condition (47) is

$$\frac{c^4s}{(48-c^4)s+48}\le s,$$

and it is satisfied for $1<c=c_1=c_2=c_3=c_4<(48)^{1/4}$. Thus. we can use $\tau=\tau_1=\tau_2=\tau_3=\tau_4=2.6<c$ and $\psi=0$ for (25) and (31). The formulas (32), (37) and (38) yield

$$\lambda_1(s)=\frac{\tau^3s^3}{32(s+1)^3}\cdot\nu_1(s),\quad \lambda_2(s)=\frac{\tau^3s^3}{36}\cdot\nu_2(s)$$
$$\lambda_3(s)=\frac{\tau^3s^3}{12(s+1)^3}\cdot\nu_3(s),\quad \lambda_4(s)=\frac{\tau^3s^3}{8}\cdot\nu_4(s).$$

Since we have $\gamma_{i,j}\hat{\alpha}_j(s)=\hat{\sigma}_{i,j}(s)$ for $\gamma_{i,j}\ge 0$, $i,j=1,2,...,4$, we can render the functions λ_i constant by making use of the functions ν_i as explained in Subsection IV-B. The choice (42) with $g_1=g_4=3$ and $g_2=g_3=1$ fulfills (33)-(36) and gives

$$\lambda_1(s)=\frac{3\tau^3}{32\cdot 27},\quad \lambda_2(s)=\frac{\tau^3}{36\cdot 8}$$
$$\lambda_3(s)=\frac{\tau^3}{12\cdot 8},\quad \lambda_4(s)=\frac{3\tau^3}{8\cdot 64}.$$

By virtue of Theorems 1 and 3, an iISS Lyapunov function of Σ is given by (18). Therefore, the network Σ is iISS.

VIII. CONCLUDING REMARKS

In this paper a constructive solution to the stability problem of iISS networks has been presented. A new theoretical approach had to be found since some central tools that have been developed to successfully analyze ISS networks are not directly applicable under the information only about supply rates of mere iISS subsystems[5]. The approach taken here is based on the sum-type Lyapunov function via graph covers with arc and vertex weights. In contrast to the previously available ISS theory based on the fact that suitably rescaled individual Lyapunov functions could be combined via maximization to yield a composite ISS Lyapunov function, here we have constructed a composite iISS Lyapunov function essentially as a sum of rescaled individual iISS Lyapunov functions. Notably, the sufficient condition for allowing such

[5]See Section I and Remark 2 for details.

a construction is, as in the ISS case, a small-gain type condition: All (suitably defined) cycles in an appropriate notion of weighted interconnection graph have to be contractions. This condition imposes that in each cycle there must be a "more stable" subsystem that makes up for the "less stable" subsystems in the cycle in a quantifiable way. The "less stable" subsystems include iISS systems which are not ISS. We also point out that this construction is applicable also to networks consisting entirely of ISS subsystems, where our construction yields a composite ISS Lyapunov function explicitly.

This paper has presented two different, but similar stability criteria corresponding to two different formulations of subsystems given by (10) and (54). The former formulation results in a small-gain condition required to hold for all cycles simultaneously, while the latter leads to a small-gain condition imposed on all cycles in a decoupled way. In other words, the former requires a small-gain condition to hold for all cycles resulting from a graph decomposition. The two criteria are qualitatively the same. To be precise in a quantitative sense, for the maximization formulation of supply rates (54), this paper has demonstrated that the developed stability criterion exactly reduces to the ones in [18]–[21] when subsystems are ISS. Apart from going beyond ISS, the constructive nature of the results in this paper is unique. The Ω-path approach based on the max-type composite Lyapunov function pursued in [17], [21] is constructive only in the maximization formulation of supply rates with help of the operator Q in [8]. This paper has presented Lyapunov functions explicitly in both maximization and summation formulations of supply rates. In the summation formulation, the construction procedure is not fully automatic since we have to choose J_U which influences the small-gain condition.

There are a lot of interesting issues which are not addressed in this paper. For example, necessary conditions of the stability of iISS networks are investigated in [33], [34], [39] where small-gain-type conditions, matrix-like conditions and the allowable number of non-ISS subsystems are discussed. An attempt to link the function Λ to the Ω-path was made in [25]. The equivalence between the small-gain condition in this paper and a matrix-like condition for the summation formulation of supply rates was discussed for a special class of systems in [39].

Appendix

A. Proof of Proposition 1

(a): Suppose that (3) holds. If $\gamma \in \mathcal{K}$, the property $\gamma^{\ominus} \circ \gamma = \mathbf{Id}$ yields $\omega(s) \leq \gamma^{\ominus}(s)$ for $s \in \mathbb{R}_+$. Applying $\gamma^{\ominus} \circ \gamma = \mathbf{Id}$ to this again, we obtain (4). Next we assume that $\gamma = \zeta^{\ominus}$ holds for some $\zeta \in \mathcal{K}$. Property (3) guarantees that $\zeta^{\ominus} \circ \omega(s)$ is finite for finite s. Thus, $\zeta \circ \zeta^{\ominus} \circ \omega(s) = \omega(s)$ holds for $s \in \mathbb{R}_+$. From (3) we obtain $\omega(s) \leq \zeta(s)$ for $s \in \mathbb{R}_+$. The property $\zeta \circ \zeta^{\ominus}(s) \leq s$ for $s \in \mathbb{R}_+$ yields (4). The converse is proved by switching γ and ω. **(b):** For $\omega \notin \mathcal{K} \cup \{\zeta^{\ominus} : \zeta \in \mathcal{K}\}$ which is not covered by **(a)**, the converse by switching in the above cannot be used. If $\gamma = \zeta^{\ominus}$ holds for $\zeta \in \mathcal{K}$, the property $\zeta^{\ominus} \circ \zeta = \mathbf{Id}$ guarantees (4) to imply (3). Next, suppose that $\gamma \in \mathcal{K}$. Property (3) holds automatically for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$. Assume $s < \lim_{\tau\to\infty} \gamma(\tau)$. Since $\gamma \circ \gamma^{\ominus}(s) = s$ holds for $s < \lim_{\tau\to\infty} \gamma(\tau)$, we obtain $\omega(s) \leq \gamma^{\ominus}(s)$ from (4). The assumption $s < \lim_{\tau\to\infty} \gamma(\tau)$ further implies $\gamma \circ \omega(s) \leq s$.

B. Proof of Lemma 1

Suppose that (23), (24) and (25) are satisfied. Let $\mathcal{CP}(i,j)$ denote the set of all paths and cycles from vertex j to vertex i of G, and define

$$\tilde{F}_{i,j}(s) = \max_{u \in \mathcal{CP}(i,j)} \hat{\sigma}_{u(1),u(2)} \circ \bigodot_{i=2}^{|u|} \underline{\alpha}_{u(i)}^{-1} \circ \overline{\alpha}_{u(i)} \circ \hat{\alpha}_{u(i)}^{\ominus} \circ \tau_{u(i)} \hat{\sigma}_{u(i),u(i+1)}(s), \quad \forall i,j = 1,2,...,n.$$

If $\mathcal{CP}(i,j) = \emptyset$ holds for a pair $i \neq j$, we replace the identically zero function $\hat{\sigma}_{i,j}$ by a new function $\hat{\sigma}_{i,j} \in \mathcal{K}\backslash\mathcal{K}_\infty$ for which (24) remains satisfied. Note that (24) can be always achieved by choosing sufficiently small $\hat{\sigma}_{i,j}$, and that such a new function satisfies (23). We compute the functions $\tilde{F}_{i,j}$ with the new $\hat{\sigma}_{i,j}$. If the set of computed $\tilde{F}_{i,j}$ satisfies (26)-(29) for the new $\hat{\sigma}_{i,j}$, the set also fulfills (26)-(29) for the original $\hat{\sigma}_{i,j}$. Notice that $\mathcal{CP}(k,k) \neq \emptyset$ holds for all k if $\mathcal{CP}(i,j) \neq \emptyset$ holds for all pairs $i \neq j$. Thus, in the remainder of this proof, we assume $\mathcal{CP}(i,j) \neq \emptyset$ for all $i,j = 1,2,...,n$. Due to (23) we have $\hat{\sigma}_{i,j} \circ \underline{\alpha}_j^{-1} \circ \overline{\alpha}_j \circ \hat{\alpha}_j^{\ominus}(s) < \infty$ and $\hat{\sigma}_{i,j} \circ \underline{\alpha}_j^{-1} \circ \overline{\alpha}_j \circ \hat{\alpha}_j^{\ominus} \in \mathcal{J} \cup \{0\}$. Thus, we have $F_{i,j} \in \mathcal{J}$ for $i,j = 1,2,...,n$. For each $j = 1,2,...,n$, the property

$$\lim_{s\to\infty} \hat{\alpha}_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} \tilde{F}_{i,j}(s) < \infty$$

holds since assumption (23) implies

$$\lim_{s\to\infty} \hat{\sigma}_{i,j} \circ \underline{\alpha}_j^{-1} \circ \overline{\alpha}_j \circ \hat{\alpha}_j^{\ominus}(s) < \infty \ \vee \ \lim_{s\to\infty} \hat{\sigma}_{i,j}(s) = \lim_{s\to\infty} \hat{\alpha}_j(s) = \infty$$

for every pair (i,j). By virtue of (24), (25) and the definition of $\tilde{F}_{i,i}$, we have

$$\underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ c_i \tilde{F}_{i,i}(s) \leq s, \ \forall s \in \mathbb{R}_+, \qquad i = 1,2,...,n.$$

Let $\mathcal{W}(i,j)$ denote the set of all walks whose last vertex and first vertex are i and j, respectively. The length of a walk can be arbitrarily large. Define

$$\bar{F}_{i,j}(s) = \sup_{w \in \mathcal{W}(i,j)} \hat{\sigma}_{w(1),w(2)} \circ \bigodot_{i=2}^{|w|} \underline{\alpha}_{w(i)}^{-1} \circ \overline{\alpha}_{w(i)} \circ \hat{\alpha}_{w(i)}^{\ominus} \circ \tau_{w(i)} \hat{\sigma}_{w(i),w(i+1)}(s), \quad \forall i,j = 1,2,...,n.$$

The property

$$\underline{\alpha}_{w(1)}^{-1} \circ \overline{\alpha}_{w(1)} \circ \hat{\alpha}_{w(1)}^{\ominus} \circ \tau_{w(1)} \bar{F}_{w(1),w(1)}(s) \leq s, \ \forall s \in \mathbb{R}_+ \tag{69}$$

holds along each closed walk $w \in \mathcal{W}(i,i)$ since (24) and (25) are assumed for all cycles in G. Recall Proposition 1.

The supremum in the definition of $\bar{F}_{i,j}$ can be replaced by a maximum, and we have $\bar{F}_{i,j} = \tilde{F}_{i,j}$. We can also verify

$$\bar{F}_{i,j}(s) = \max\left\{ \max_{\substack{1 \le q \le n \\ q \neq i, q \neq j}} \bar{F}_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q \bar{F}_{q,j}(s),\ \hat{\sigma}_{i,j}(s) \right\}.$$

Therefore, we arrive at (26)-(29) for $F_{i,j} = \tilde{F}_{i,j}$, $i,j = 1,2,...,n$.

C. Proof of Theorem 2

First, properties $\overline{\lambda}_i \in \mathcal{J}$, (32), (33), (35) and (37) imply (12) and (13), and the non-decreasing property of λ_i and (17) follow. Since $\lim_{s\to\infty} \hat{\alpha}_i(s) = \infty$ is equivalent to $\lim_{s\to\infty} \alpha_i(s) = \infty$, properties (29) and (34) imply (14). Secondly, from (26) and (27) it follows that

$$F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q \hat{\sigma}_{q,j}(s) \le F_{i,j}(s), \quad i,q,j = 1,2,...,n,\ i \neq q,\ q \neq j. \tag{70}$$

Choose an arbitrary cycle graph $U \in \mathcal{C}(G)$ and suppose that $J_U \neq 0$. Define

$$\tilde{\nu}_{u(j)}(s) = \prod_{q\in\mathcal{V}(G)-\mathcal{V}(U)} \left[F_{q,u(j)}(\underline{\alpha}_{u(j)}^{-1}(s))\right]^{\psi+1}, \quad j=1,2,...,|U|. \tag{71}$$

In the case of $\mathcal{V}(G) = \mathcal{V}(U)$, the above definition is intended as $\tilde{\nu}_{u(j)}(s) \equiv 1$. Since $0 < F_{i,j}(s) < \infty$ holds for $s \in (0,\infty)$, we have $0 < \tilde{\nu}_i(s) < \infty$ for $s \in (0,\infty)$. The property

$$\tilde{\nu}_{u(j)} \circ \overline{\alpha}_{u(j)} \circ \hat{\alpha}_{u(j)}^{\ominus} \circ \tau_{u(j)} \hat{\sigma}_{u(j),u(j+1)}(s) \le \tilde{\nu}_{u(j+1)} \circ \underline{\alpha}_{u(j+1)}(s),\ \forall s \in \mathbb{R}_+,\ j=1,2,...,|U|$$

is implied by (70). Thus, due to (31), for the cycle graph U, replacing ν_i with $\tilde{\nu}_i$, we have (33), (34) and (53) in Theorem 3 for cycle networks. It can be verified that the functions λ_i, $i = 1,2,...,n$ given in (37) are in the form of

$$\lambda_{u(j)}(s) = \overline{\lambda}_{u(j)}(s)\nu_{u(j)}(s) = \tilde{\lambda}_{u(j)}(s)\tilde{\nu}_{u(j)}(s)\nu_{u(j)}(s), \quad j = 1,2,...,|U|$$

$$\tilde{\lambda}_{u(j)}(s) = \left[\frac{1}{\tau_{u(j)}} \hat{\alpha}_{u(j)}(\overline{\alpha}_{u(j)}^{-1}(s))\right]^{\psi} \cdot \prod_{q\in\mathcal{V}(U)-\{u(j)\}} \left[F_{q,u(j)}(\underline{\alpha}_{u(j)}^{-1}(s))\right]^{\psi+1},\ j = 1,2,...,|U|$$

for $\tilde{\nu}_{u(j)}$ of each cycle graph U. Property (53) in Theorem 3 is guaranteed by (36). Applying Theorem 3 to all cycle graphs $U \in \mathcal{C}(G)$ satisfying $J_U \neq 0$, we obtain $\hat{b}_i > 0$, $i = 1,2,...,n$, such that

$$\sum_{i=1}^{|U|} \lambda_{u(i)}(s_{u(i)}) \Big[-(\mathbf{Id}+\hat{\delta}_{u(i)})^{-1} \circ \hat{\alpha}_{u(i)} \circ \overline{\alpha}_{u(i)}^{-1}(s_{u(i)}) + \hat{\sigma}_{u(i),u(i+1)} \circ \underline{\alpha}_{u(i+1)}^{-1}(s_{u(i+1)}) \Big] \le 0$$

holds with $\hat{\delta}_i(s) = \hat{b}_i s$ for all such U. Next, suppose that T is a path graph, i.e., $T \in \mathcal{P}(G)$ consisting of $t = (t(1), t(2), ..., t(|T|+1))$, such that $J_T \neq 0$. Using the fictitious arc defined with $F_{t(|T|+1),t(1)}$, we define the fictitious cycle

$$\begin{aligned} u &= (u(1), u(2), ..., u(|T|+1), u(1)) \\ &= (t(1), t(2), ..., t(|T|+1), t(1)) \end{aligned} \tag{72}$$

to define a cycle graph U, and we repeat the above argument for cycle graphs to obtain

$$\sum_{i=1}^{|U|-1} \lambda_{u(i)}(s_{u(i)}) \Big[-(\mathbf{Id}+\hat{\delta}_{u(i)})^{-1} \circ \hat{\alpha}_{u(i)} \circ \overline{\alpha}_{u(i)}^{-1}(s_{u(i)}) + \hat{\sigma}_{u(i),u(i+1)} \circ \underline{\alpha}_{u(i+1)}^{-1}(s_{u(i+1)}) \Big] + \lambda_{u(|U|)}(s_{u(|U|)}) \cdot \Big[-(\mathbf{Id}+\hat{\delta}_{u(|U|)})^{-1} \circ \hat{\alpha}_{u(|U|)} \circ \overline{\alpha}_{u(|U|)}^{-1}(s_{u(|U|)}) \Big] \le 0.$$

By (20)-(22) we have

$$\begin{aligned} &\Lambda(\mathbf{s})^T \Big[-D^{-1} \circ A(\mathbf{s}) + S(\mathbf{s}) \Big] = \\ &\sum_{U\in\mathcal{C}(G)} J_U \sum_{i=1}^{|U|} \lambda_{u(i)}(s_{u(i)}) \Big[-(\mathbf{Id}+\hat{\delta}_{u(i)})^{-1} \circ \hat{\alpha}_{u(i)} \circ \overline{\alpha}_{u(i)}^{-1}(s_{u(i)}) + \hat{\sigma}_{u(i),u(i+1)} \circ \underline{\alpha}_{u(i+1)}^{-1}(s_{u(i+1)}) \Big] \\ &+ \sum_{T\in\mathcal{P}(G)} J_T \Bigg\{ \sum_{i=1}^{|T|} \lambda_{t(i)}(s_{t(i)}) \Big[-(\mathbf{Id}+\hat{\delta}_{t(i)})^{-1} \circ \hat{\alpha}_{t(i)} \circ \overline{\alpha}_{t(i)}^{-1}(s_{t(i)}) + \hat{\sigma}_{t(i),t(i+1)} \circ \underline{\alpha}_{t(i+1)}^{-1}(s_{t(i+1)}) \Big] \\ &\quad + J_T \lambda_{t(|T|+1)}(s_{t(|T|+1)}) \Big[-(\mathbf{Id}+\hat{\delta}_{t(|T|+1)})^{-1} \circ \hat{\alpha}_{t(|T|+1)} \circ \overline{\alpha}_{t(|T|+1)}^{-1}(s_{t(|T|+1)}) \Big] \Bigg\} \\ &+ \sum_{Y\in\mathcal{V}(G)} J_Y \lambda_{y(1)}(s_{y(1)}) \Big\{ -(\mathbf{Id}+\hat{\delta}_{y(1)})^{-1} \circ \hat{\alpha}_{y(1)} \circ \overline{\alpha}_{y(1)}^{-1}(s_{y(1)}) \Big\} \le 0 \end{aligned}$$

with $\delta_i(s) = \hat{\delta}_i(d_i s)/d_i = \hat{b}_i s$ since $(\mathbf{Id}+\delta_i)^{-1}(s) = (\mathbf{Id}+\hat{\delta}_i)^{-1}(d_i s)/d_i$. We arrive at (15).

D. Proof of Proposition 2

Define

$$h(x,r) = \sum_{i=1}^{n} \lambda_i(V_i(x_i)) \left\{ -\alpha_i(|x_i|) + \sum_{j=1}^{n} \sigma_{i,j}(|x_j|) + \kappa_i(|r|) \right\}, \tag{73}$$

where $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions. Suppose that there exist a $k \in \{1,2,...,n\}$ such that

$$\lim_{s\to\infty} \alpha_k(s) < \infty \ \wedge \ \lim_{s\to\infty} \sum_{i=1}^{n} \sigma_{i,k}(s) = \infty. \tag{74}$$

If $h(x,0) \le 0$ holds for all $x \in \mathbb{R}^N$, definition (73) yields

$$\liminf_{s\to\infty} \lambda_k(s) = \infty. \tag{75}$$

Since we have

$$\begin{aligned} &h(x,r) \\ &\geq \sum_{i=1,i\neq k}^{n} \lambda_i(V_i(x_i)) \left\{ -\alpha_i(|x_i|) + \sum_{j=1}^{n} \sigma_{i,j}(|x_j|) + \kappa_i(|r|) \right\} \\ &\quad + \lambda_k(V_k(x_k)) \left\{ \kappa_k(|r|) - \alpha_k(|x_k|) \right\}, \end{aligned} \tag{76}$$

properties (74) and (75) imply that

$$h(x,r) \leq \beta(|r|), \quad \forall (x,r) \in \mathbb{R}^N \times \mathbb{R}^M \tag{77}$$

cannot be achieved by any continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. Using the technique presented in [34], we can construct a network Σ and V_i's satisfying (9) and (10) such that $\dot{V} = h(x,r)$, $\forall (x,r) \in [\epsilon, \infty)^N \times \mathbb{R}^M$ is satisfied along the trajectories of Σ for an arbitrarily given constant $\epsilon > 0$. Since (77) is not achievable, no function V in the form of (2) can be an iISS Lyapunov function, where the functions λ_i' are the derivatives of the functions W_i in (2).

E. Proof of Lemma 2

The claim is proved by defining, for each $i = 1, 2, ..., n$,

$$\begin{aligned} &F_{i,j}(s) = \hat{\sigma}_{i,j}(s), \qquad j = (i \mod n) + 1 \\ &F_{i,j} = F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \hat{\alpha}_q^{\ominus} \circ \tau_q \hat{\sigma}_{q,j}(s) \\ &\qquad\qquad i+2 \leq h \leq i+n, \quad q = (h-2 \mod n) + 1 \\ &\qquad\qquad j = (h-1 \mod n) + 1, \end{aligned}$$

and verifying (48)-(52) and (29).

F. Proof of Theorem 3

Properties $\overline{\lambda}_i \in \mathcal{J}$, (32), (33), (35) and (37) yield (12), (13) the non-decreasing property of λ_i and (17). Property (14) is implied by (29), (32), (34) and (37). Let

$$\begin{aligned} \theta_i(s) &= \overline{\alpha}_i \circ \hat{\alpha}_i^{\ominus} \circ \tau_i \hat{\sigma}_{i,j}(s) \\ &= \begin{cases} \overline{\alpha}_i \circ \hat{\alpha}_i^{-1} \circ \tau_i \hat{\sigma}_{i,j}(s) & \text{if } \lim_{\iota \to \infty} \hat{\alpha}_i(\iota) > \tau_i \hat{\sigma}_{i,j}(s) \\ \lim_{\iota \to \infty} \overline{\alpha}_i(\iota) & \text{otherwise} \end{cases}, \end{aligned}$$

for $i = 1, 2, ..., n$ and $j = (i \mod n) + 1$. By definition we have

$$\begin{aligned} \hat{\alpha}_i \circ \overline{\alpha}_i^{-1} \circ \theta_i(s) &= \begin{cases} \tau_i \hat{\sigma}_{i,j}(s) & \text{if } \lim_{\iota \to \infty} \hat{\alpha}_i(\iota) > \tau_i \hat{\sigma}_{i,j}(s) \\ \lim_{\iota \to \infty} \hat{\alpha}_i(\iota) & \text{otherwise} \end{cases} \\ &\leq \tau_i \hat{\sigma}_{i,j}(s). \end{aligned} \tag{78}$$

Define $b_i, \epsilon_i > 0$ by

$$b_i = \frac{\tau_i}{\epsilon_i(\tau_i - 1) + 1} - 1, \quad \epsilon_i = \frac{\tau_i}{c_i}, \quad i = 1, 2, ..., n.$$

Combining the two cases $\hat{\alpha}_i(\overline{\alpha}_i^{-1}(s_i)) > \tau_i \hat{\sigma}_{i,j}(\underline{\alpha}_i^{-1}(s_j))$ and $\hat{\alpha}_i(\overline{\alpha}_i^{-1}(s_i)) \leq \tau_i \hat{\sigma}_{i,j}(\underline{\alpha}_i^{-1}(s_j))$ yields

$$\begin{aligned} &\lambda_i(s_i) \left(-\frac{1}{1+b_i} \hat{\alpha}_i(\overline{\alpha}_i^{-1}(s_i)) + \hat{\sigma}_{i,j}(\underline{\alpha}_i^{-1}(s_j)) \right) \\ &\quad \leq -\left(\frac{1}{1+b_i} - \frac{1}{\tau_i} \right) \lambda_i(s_i) \hat{\alpha}_i(\overline{\alpha}_i^{-1}(s_i)) \\ &\qquad + \lambda_i(\theta_i(\underline{\alpha}_i^{-1}(s_j))) \hat{\sigma}_{i,j}(\underline{\alpha}_i^{-1}(s_j)), \end{aligned}$$

since λ_i is non-decreasing. Notice that, due to (14), the value of $\lambda_i(\theta_i(s))\hat{\sigma}_{i,j}(s)$, $i = 1, 2, ..., n$, is finite for all $s \in \mathbb{R}_+$. The choice $\delta_i(s) = b_i s$ implies that (15) holds for (44) and (45) if λ_i, $i = 1, 2, ..., n$ satisfy

$$\begin{aligned} &\lambda_i(\theta_i(s))\hat{\sigma}_{i,j}(s) \leq \epsilon_j \cdot \frac{\tau_j - 1}{\tau_j} \lambda_j(\underline{\alpha}_j(s)) \hat{\alpha}_j(\overline{\alpha}_j^{-1}(\underline{\alpha}_j(s))), \\ &\qquad \forall s \in \mathbb{R}_+, \ i = 1, 2, ..., n, \ j = (i \mod n) + 1. \end{aligned} \tag{79}$$

Pick $i \in \mathcal{V}(G) = \{1, 2, ..., n\}$ and define $j = (i \mod n) + 1$. Using (32), (37) and (38), we obtain

$$\begin{aligned} &\lambda_i(\theta_i(s))\hat{\sigma}_{i,j}(s) \\ &\leq \nu_i(\theta_i(s)) \left[\hat{\sigma}_{i,j}(s)\right]^{\psi+1} \prod_{q \in \mathcal{V}(G) - \{i\}} \left[F_{q,i}(\underline{\alpha}_i^{-1}(\theta_i(s)))\right]^{\psi+1} \\ &\leq \nu_i(\theta_i(s)) \left[\hat{\sigma}_{i,j}(s)\right]^{\psi+1} \prod_{q \in \mathcal{V}(G) - \{i\}} \left[F_{q,j}(s)\right]^{\psi+1} \\ &= \nu_i(\theta_i(s)) \left[\hat{\sigma}_{i,j}(s)\right]^{\psi+1} \left[F_{j,j}(s)\right]^{\psi+1} \prod_{q \in \mathcal{V}(G) - \{i,j\}} \left[F_{q,j}(s)\right]^{\psi+1}. \end{aligned} \tag{80}$$

Here, the first and second inequalities make use of (78) and (49), respectively. We also obtain

$$\begin{aligned} &\epsilon_j \cdot \frac{\tau_j - 1}{\tau_j} \lambda_j(\underline{\alpha}_j(s)) \hat{\alpha}_j(\overline{\alpha}_j^{-1}(\underline{\alpha}_j(s))) \\ &\quad = \epsilon_j(\tau_j - 1) \cdot \nu_j(\underline{\alpha}_j(s)) \left[F_{i,j}(s)\right]^{\psi+1} \cdot \\ &\qquad \left[\frac{1}{\tau_j} \hat{\alpha}_j(\overline{\alpha}_j^{-1}(\underline{\alpha}_j(s)))\right]^{\psi+1} \prod_{q \in \mathcal{V}(G) - \{i,j\}} \left[F_{q,j}(s)\right]^{\psi+1}. \end{aligned} \tag{81}$$

From (50) and $\hat{\alpha}_i \circ \hat{\alpha}_i^{\ominus}(s) \leq s$ for $s \in \mathbb{R}_+$, we obtain

$$\left[F_{j,j}(s)\right]^{\psi+1} \leq \left(\frac{\tau_j}{c_j}\right)^{\psi+1} \left[\frac{1}{\tau_j} \hat{\alpha}_j(\overline{\alpha}_j^{-1}(\underline{\alpha}_j(s)))\right]^{\psi+1}, \forall s \in \mathbb{R}_+.$$

From (31) it follows that $(\tau_j/c_j)^{\psi+1} \leq \epsilon_j(\tau_j - 1)$. With the help of (25), (48) and (53) applied to (80) and (81), we arrive at (79) for each $i = 1, 2, ..., n$. This completes the proof.

G. Proof of Lemma 3

Removing hats from $\hat{\alpha}_i$ and $\hat{\sigma}_{i,j}$ in Lemma 2, we obtain Lemma 3.

H. Proof of Theorem 4

From the definition (55) we obtain

$$\begin{aligned}
&\Lambda(\mathbf{s})^T\left[-D^{-1}\circ A(\mathbf{s})+S(\mathbf{s})\right]= \\
&\sum_{U\in\mathcal{C}(G)}K(U,\mathbf{s})\sum_{i=1}^{|U|}\lambda_{u(i)}(s_{u(i)})\Big[-(\mathbf{Id}+\delta_{u(i)})^{-1}\circ\alpha_{u(i)}(s_{u(i)}) \\
&\qquad\qquad +\sigma_{u(i),u(i+1)}(s_{u(i+1)})\Big] \\
&+\sum_{T\in\mathcal{P}(G)}K(T,\mathbf{s})\left\{\sum_{i=1}^{|T|}\lambda_{t(i)}(s_{t(i)})\Big[-(\mathbf{Id}+\delta_{t(i)})^{-1}\circ\alpha_{t(i)}(s_{t(i)})\right. \\
&\qquad\qquad +\sigma_{t(i),t(i+1)}(s_{t(i+1)})\Big] \\
&+K(T,\mathbf{s})\lambda_{t(|T|+1)}(s_{t(|T|+1)})\Big[-(\mathbf{Id}+\delta_{t(|T|+1)})^{-1}\circ \\
&\qquad\qquad \left.\alpha_{t(|T|+1)}(s_{t(|T|+1)})\Big]\right\} \\
&+\sum_{Y\in\mathcal{V}(G)}K(Y,\mathbf{s})\lambda_{y(1)}(s_{y(1)})\left\{-(\mathbf{Id}+\delta_{y(1)})^{-1}\circ\alpha_{y(1)}(s_{y(1)})\right\}
\end{aligned} \tag{82}$$

for all $\mathbf{s}=[s_1,s_3,...,s_n]^T\in\mathbb{R}_+^n$, where the mappings $K(Z,\mathbf{s}):\mathcal{CPV}(G)\times\mathbb{R}_+^n\to\{0,1\}$ are boolean-valued and satisfy[6]

$$G=\bigcup_{\mathbf{s}\in\mathbb{R}_+^n}\ \bigcup_{W(\mathbf{s})\in\{Z\in\mathcal{CPV}(G):\ K(Z,\mathbf{s})=1\}} W(\mathbf{s})\ .$$

Each $K(Z,\mathbf{s})$ depends on $\mathbf{s}$ since $S(\mathbf{s})$ varies with $\mathbf{s}$. In other words, for each $\mathbf{s}\in\mathbb{R}_+^n$, only one of $\{\sigma_{i,j}(s_j):j=1,2,...,n\}$ is attained in the maximization (55). As in the case of summation supply rates, we apply the single cycle formula to each cycle graph $U\in\mathcal{C}(G)$ to compute $\tilde{\lambda}_{u(j)}$ for $j\in\mathcal{V}(U)$ achieving

$$\begin{aligned}
&\sum_{i=1}^{|U|}\tilde{\lambda}_{u(i)}(s_{u(i)})\Big[-(\mathbf{Id}+\delta_{u(i)})^{-1}\circ\alpha_{u(i)}\circ\overline{\alpha}_{u(i)}^{-1}(s_{u(i)}) \\
&\qquad\qquad +\sigma_{u(i),u(i+1)}\circ\underline{\alpha}_{u(i+1)}^{-1}(s_{u(i+1)})\Big]\le 0.
\end{aligned}$$

For each path graph $T\in\mathcal{P}(G)$, we can use $F_{t(|T|+1),t(1)}$ to define a cycle graph U as in (72) to which we apply the single cycle formula to compute $\tilde{\lambda}_{u(j)}$. Finally, we obtain (12)-(15) and (17) for λ_i'given by (37), (38) and (61) by matching $\tilde{\lambda}_j$ computed for different cycles to each other with the help of $\tilde{\nu}_j$ as in the proof of Theorem 2.

I. Proof of Lemma 4

Suppose that (63) is true. Then for each $\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\}$, there exists a $k\in\{1,2,...,n\}$ such that $[D\circ S(\mathbf{s})]_k<[A(\mathbf{s})]_k=\alpha_k\circ\overline{\alpha}_k^{-1}(s_k)$. By $\overline{\alpha}_k\in\mathcal{K}_\infty$ and $\alpha_k^\ominus$ which is continuous and strictly increasing on $[0,\lim_{\tau\to\infty}\alpha_k(\tau))$, we have $\overline{\alpha}_k\circ\alpha_k^\ominus\circ[D\circ S(\mathbf{s})]_k<\overline{\alpha}_k\circ\alpha_k^\ominus\circ\alpha_k\circ\overline{\alpha}_k^{-1}(s_k)=s_k$. We arrive at (64). Next, suppose that (64) is true. Then for each $\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\}$, there exists a $k\in\{1,2,...,n\}$ such that $\alpha_k^\ominus\circ[D\circ S(\mathbf{s})]_k<\overline{\alpha}_k^{-1}(s_k)$. This property together with $\overline{\alpha}_k\in\mathcal{K}_\infty$ implies $\lim_{\tau\to\infty}\alpha_k(\tau)>[D\circ S(\mathbf{s})]_k$. The definition of $\alpha_k^\ominus$ yields $\alpha_k\circ\alpha_k^\ominus\circ[D\circ S(\mathbf{s})]_k=[D\circ S(\mathbf{s})]_k<\alpha_k\circ\overline{\alpha}_k^{-1}(s_k)$ from which (63) follows. The equivalence between (63) and (65) is straightforward from $\mathbf{Id}+\delta_i\in\mathcal{K}_\infty$.

J. Proof of Proposition 3

The map $M:=N\circ D\circ S:\overline{\mathbb{R}}_+^n\to\overline{\mathbb{R}}_+^n$ is monotone on $\overline{\mathbb{R}}_+^n$, i.e., $x\le y$ implies $M(x)\le M(y)$. To prove the implication that (64) entails (66) for all $U\in\mathcal{C}(G)$, the invariance of condition (66) with respect to index rotation proved by Proposition 1 allows us to follow the argument in [40, Theorem 6.4]. The converse implication is proved in [40, Theorem 6.4] for $\alpha_i\in\mathcal{K}_\infty$, $i=1,2,...,n$. To modify the monotonicity approach for addressing $\alpha_i\in\mathcal{K}\setminus\mathcal{K}_\infty$ in the converse, define a directed graph $G(\mathbf{s})$ associated with the operator $M(\mathbf{s})$ for each $\mathbf{s}\in\mathbb{R}_+^n$. The pair (i,j) is an element of the arc set $\mathcal{A}(G(\mathbf{s}))$ if and only if $\max_{k\in\{1,2,...,n\}}\sigma_{i,k}(\mathbf{s})=\sigma_{i,j}(\mathbf{s})\neq 0$. The vertex set is defined as $\mathcal{V}(G(\mathbf{s})):=\mathcal{V}(G)=\{1,2,...,n\}$. We have $\mathcal{C}(G(\mathbf{s}))\subseteq\mathcal{C}(G)$, $\mathcal{P}(G(\mathbf{s}))\subseteq\mathcal{P}(G)$ and $\mathcal{I}(G)\subseteq\mathcal{I}(G(\mathbf{s}))$. For an arbitrary $U\in\mathcal{I}(G(\mathbf{s}))$, from $M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{s}_{\mathcal{V}(U)})=0$ it follows that

$$U\in\mathcal{I}(G(\mathbf{s}))\ \wedge\ \mathbf{s}_{\mathcal{V}(U)}\neq 0\ \Rightarrow\ M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{s}_{\mathcal{V}(U)})\not\geq\mathbf{s}_{\mathcal{V}(U)}. \tag{83}$$

Next, to take account of path graphs in $G(\mathbf{s})$, we define

$$\mathcal{R}(G(\mathbf{s})):=\left\{R:R=\bigcup_{T\in\mathcal{T}_i(\mathbf{s})}T,\quad i\in\mathcal{V}(G)\right\},$$

where $\mathcal{T}_i(\mathbf{s}):=\{T\in\mathcal{P}(G(\mathbf{s})):t(|T|+1)=i,\ (t(|T|+1),j)\notin\mathcal{A}(G(\mathbf{s})),\ \forall j\in\mathcal{V}(G)\}$. We have $\{\mathcal{P}(G(\mathbf{s}))=\emptyset\Leftrightarrow\mathcal{R}(G(\mathbf{s}))=\emptyset\}$. For an arbitrary graph $U\in\mathcal{R}(G(\mathbf{s}))$, we have $s_{t(k)}>0$ for $k=2,3,...,|T|+1$ for all $T\in\mathcal{T}_i(\mathbf{s})$ composing U. Due to the maximization in M implying exclusive connection, $U\in\mathcal{R}(G(\mathbf{s}))$ implies $M_{t(|T|+1),\mathcal{V}(U)}(\mathbf{s}_{\mathcal{V}(U)})=0<s_{t(|T|+1)}$. Hence,

$$U\in\mathcal{R}(G(\mathbf{s}))\ \Rightarrow\ M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{s}_{\mathcal{V}(U)})\not\geq\mathbf{s}_{\mathcal{V}(U)}. \tag{84}$$

To consider cycle graphs in $G(\mathbf{s})$, we define

$$\mathcal{B}(G(\mathbf{s})):=\left\{B:B=W\cup\left(\bigcup_{T\in\mathcal{P}_W(\mathbf{s})}T\right),\ W\in\mathcal{C}(G(\mathbf{s}))\right\}.$$

where $\mathcal{P}_W(\mathbf{s}):=\{T\in\mathcal{P}(G(\mathbf{s})):\exists i\in\mathcal{V}(W)\text{ s.t. }t(|T|+1)=w(i)\}$ and $\mathcal{P}_\emptyset(\mathbf{s}):=\emptyset$. We have $\{\mathcal{C}(G(\mathbf{s}))=\emptyset\Leftrightarrow\mathcal{B}(G(\mathbf{s}))=\emptyset\}$. For each $B\in\mathcal{B}(G(\mathbf{s}))$, there is a unique $W\in\mathcal{C}(G(\mathbf{s}))$ such that $\mathcal{A}(W)\subseteq\mathcal{A}(B)$. We assume that (66) holds for all cycles of $\mathcal{C}(G)$. Suppose that a vector $\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\}$ is given and fixed. Consider a graph $U\in\mathcal{B}(G(\mathbf{s}))$ containing a cycle graph Q, and set $L=\#\mathcal{V}(U)$. Then we have $s_{q(i)}>0$ for $i=1,2,...,|Q|$. Take the maximization of S in M for the given $\mathbf{s}$ and define $Z_{\mathbf{s}}:\overline{\mathbb{R}}_+^L\to\overline{\mathbb{R}}_+^L$ by

$$Z_{\mathbf{s}}=[N\circ D]_{\mathcal{V}(U),\mathcal{V}(U)}\circ\begin{bmatrix}\sigma_{a(1),b(1)}\circ\underline{\alpha}_{b(1)}^{-1}\\ \vdots\\ \sigma_{a(L),b(L)}\circ\underline{\alpha}_{b(L)}^{-1}\end{bmatrix}$$

[6]There was a typo in the next equation on p.1205 of the TAC publication.

with appropriate $a(i)$, $b(i) \in \mathcal{V}(U)$, $i = 1, 2, ..., L$ such that $Z_{\mathbf{s}}(\mathbf{s}) = M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{s})$. This operator $Z_{\mathbf{s}}$ is different from $M_{\mathcal{V}(U),\mathcal{V}(U)}$ since it does not involve maximization any more. Let $\mathbf{p} = \mathbf{s}$, and temporarily assume that $M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{p}_{\mathcal{V}(U)}) \geq \mathbf{p}_{\mathcal{V}(U)}$ is true. Due to the monotonicity of $Z_{\mathbf{s}}$ on $\overline{\mathbb{R}}_+^L$, we have $Z_{\mathbf{s}}^{k|Q|}(\mathbf{p}_{\mathcal{V}(U)}) \geq \mathbf{p}_{\mathcal{V}(U)}$ for all integers $k \geq 1$. Therefore, we obtain

$$\begin{aligned} 0 &< p_{q(1)} \\ &\leq \left[Z_{\mathbf{s}}^{k|Q|}(\mathbf{p}_{\mathcal{V}(U)}) \right]_{q(1)} \\ &= \left[\underline{\alpha}_{q(1)} \circ \gamma_{q(1),q(2)} \circ \gamma_{q(2),q(3)} \circ \cdots \gamma_{q(|Q|),q(1)} \circ \underline{\alpha}_{q(1)}^{-1} \right]^k (p_{q(1)}), \end{aligned} \tag{85}$$

where $\gamma_{i,j} = \underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \alpha_i^{\ominus} \circ (\mathbf{Id} + \delta_i) \circ \sigma_{i,j}$. By virtue of (66) and Proposition 1, as k tends to infinity, the right hand side of (85) decreases to zero. This contradicts $p_{q(1)} > 0$ fixed. Hence

$$U \in \mathcal{B}(G(\mathbf{s})) \;\Rightarrow\; M_{\mathcal{V}(U),\mathcal{V}(U)}(\mathbf{s}_{\mathcal{V}(U)}) \ngeq \mathbf{s}_{\mathcal{V}(U)}. \tag{86}$$

must hold. By virtue of the maximization in $S(\mathbf{s})$ implying exclusive connection, we obtain

$$\{U \in \mathcal{I}(G(\mathbf{s})) : \mathbf{s}_{\mathcal{V}(U)} \neq 0\} \cup \mathcal{R}(G(\mathbf{s})) \cup \mathcal{B}(G(\mathbf{s})) \neq \emptyset$$

and

$$\begin{aligned} U, W \in \mathcal{I}(G(\mathbf{s})) \cup \mathcal{R}(G(\mathbf{s})) \cup \mathcal{B}(G(\mathbf{s})),\ i \in \mathcal{V}(U),\ i \in \mathcal{V}(W) \\ \Rightarrow\ U = W \end{aligned}$$

for each $\mathbf{s} \in \mathbb{R}_+^n \setminus \{0\}$. Therefore, properties (83), (84) and (86) guarantee (64).

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