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Utility of iISS in Composing Lyapunov Functions for Interconnections

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Abstract: Decomposition of a system into smaller components sometimes allows us to analyze and design the system effectively based on properties of the components. The notion of input-to-state stability (ISS) has been widely used to characterize components that refuse linear-like properties. It is, however, still restrictive, and it cannot cover a lot of saturation mechanisms which often arise in practical systems. The notion of integral input-to-state stability (iISS) is a way to remove the limitation of ISS. This paper collects and illustrates some recent advances in the framework of iISS that allows us to broaden the class of nonlinearities we can address in analysis and design of interconnected systems by making use of Lyapunov functions.

Keywords: Nonlinear systems; Interconnected systems; Integral input-to-state stability; Lyapunov functions.

1. INTRODUCTION

In analyzing and designing complex or large-scale systems, bottom-up approaches are sometimes useful. If smaller modules called subsystems enjoy some useful properties individually, the aggregation of those properties allows us to deduce a desirable property of the original large system. Since the early days of control engineering, the fact that a loop gain of less than unity ensures stability of a feedback loop consisting of two subsystems is widely recognized. The loop gain is an aggregate of gains of individual subsystems. This idea was formulated mathematically into the small-gain theorem in Zames [1966], and the classical small-gain theorem and its generalization have been widely recognized as indispensable tools for stability analysis and control design in both linear and nonlinear systems. An extension of the classical small-gain theorem was first made to cover a class of truly nonlinear characteristics in Hill [1991], Mareels and Hill [1992] by making use of nonlinear gains in the framework of input-output operator theory. On the other hand, relying upon Sontag's seminal work on input-to-state stability (ISS) and its various equivalent characterizations (e.g. Sontag [1989], Sontag and Wang [1995]), the ISS small-gain theorem was established in Jiang et al. [1994], which has been playing an important role in nonlinear systems analysis and design. The further development in Teel [1996] has allowed us to make use of the small-gain argument to design a class of systems with saturation. Utilization of nonlinear loop gains for interconnected ISS systems led to small-gain arguments in a general framework of monotone systems in Angeli and Sontag [2003], Enciso and Sontag [2006], just to name a few.

The essential mechanism of the small-gain theorems is simple. The magnitude of signals is never amplified by making a circuit of the feedback loop. In fact, the small-gain theorems have been proved based on signals or operators from one signal space to another. These proofs are sometimes referred to as trajectory-based or operator-theoretic approaches. Another group of approaches is Lyapunov-based. The classical and ISS small-gain theorems were interpreted in terms of construction of a Lyapunov function of a feedback loop in Hill and Moylan [1977] and Jiang et al. [1996], respectively. In many circumstances, the knowledge of a Lyapunov function of a system is more preferable than a guarantee of a single property since many properties can be extracted from a single Lyapunov function. The usefulness of Lyapunov-based approaches is more significant when we encounter systems for which we do not know how to establish a small-gain theorem using trajectory-based and operator-theoretic approaches.

The notion of ISS introduced by Sontag [1989] describes how robust a system is with respect to disturbance input. Another notion, integral input-to-state stability (iISS) proposed in Sontag [1998] characterizes a similar robustness property. In contrast to ISS, the iISS does not require bounded magnitude of the state even for an input of bounded magnitude. In applications, such unboundedness is often inevitable due to saturation and limitations of stabilizing signals. In spite of this ubiquity, the unboundedness had made the iISS property somewhat less attractive in developing bottom-up methodologies for analyzing and designing complex systems. In fact, we encounter obstacles when we try to extend the ISS small-gain theorem to iISS systems. The purpose of this paper is to briefly review those obstacles and illustrate the key ideas of some resent results to overcome those obstacles. The presented materials were developed on the basis of many preceeding important results in the literature. To try to make this paper accessible for a variety of readers in the limited space, some fundamentals are recalled without going into details of their historical background and motivation.

Notations

In this paper, the symbol $\mathbb{R}$ denotes the set of real numbers $(-\infty, \infty)$. $\overline{\mathbb{R}}$ denotes the extended real line $[-\infty, \infty]$. We also use $\mathbb{R}_+ = [0,\infty)$ and $\overline{\mathbb{R}}_+ = [0,\infty]$. For a positive integer N, $\mathbb{R}^N$ denotes the linear space over $\mathbb{R}$ of all N-tuples of real numbers. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector in $\mathbb{R}^N$. For vectors $a, b \in \mathbb{R}^N$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, N$. The negation of $a \geq b$ is denoted by $a \ngeq b$, i.e., there exists an $i \in \{1, \ldots, N\}$ such that $a_i < b_i$. The relation $a \gg b$ is defined by $a_i > b_i$ for all $i = 1, \ldots, N$. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. In this paper, a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ if it is continuous, zero at zero, and positive elsewhere. A class $\mathcal{P}$ function is of class $\mathcal{J}$ if it is non-decreasing. A class $\mathcal{J}$ function is of class $\mathcal{K}$ if it is strictly increasing. A class $\mathcal{K}$ function is of class $\mathcal{K}_\infty$ if it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in the first argument and it monotonically decreases to zero in the second argument. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. The symbol sgn denotes

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$$

for $x \in \mathbb{R}$. Global asymptotic stability of an equilibrium of a system without input is referred to GAS.

2. IISS AND ISS

Consider the system Σ described by

$$\Sigma : \ \dot{x}(t) = f(x(t), r(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^N$ is a state vector, and the function $r : \mathbb{R}_+ \to \mathbb{R}^P$ is a disturbance input which is assumed to be measurable, locally essentially bounded. The function $f : \mathbb{R}^N \times \mathbb{R}^P \to \mathbb{R}^N$ is assumed to be locally Lipschitz. In Sontag [1989], Sontag [1998] and Angeli et al. [2000], the notions of iISS and ISS are defined as follows:

Definition 1. System (1) is said to be integral input-to-state stable (iISS) with respect to r if there exist a $\mathcal{K}_\infty$ function χ, a $\mathcal{KL}$ function β and a $\mathcal{K}$ function μ such that, for any initial state $x(0) \in \mathbb{R}^N$ and any measurable, locally essentially bounded input r, the corresponding solution exists for all $t \geq 0$, and furthermore it satisfies

$$\chi(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \mu(|r(\tau)|)\, d\tau. \tag{2}$$

Definition 2. System (1) is said to be input-to-state stable (ISS) with respect to r if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial state $x(0) \in \mathbb{R}^N$ and any measurable, locally essentially bounded input r, the corresponding solution exists for all $t \geq 0$, and furthermore it satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{\tau \in [0,t]} |r(\tau)|\right). \tag{3}$$

In (3), the symbol sup denotes the essential supremum. The iISS and ISS properties can be characterized in terms of Lyapunov-like functions in the following way (see Sontag and Wang [1995], Angeli et al. [2000]) .

Definition 3. A continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called an iISS Lyapunov function for system (1) if there exist $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{4}$$

$$\frac{\partial V}{\partial x} f(x, r) \leq -\alpha(|x|) + \sigma(|r|) \tag{5}$$

hold for all $x \in \mathbb{R}^N$ and all $r \in \mathbb{R}^P$.

Definition 4. A continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called an ISS Lyapunov function for system (1) if there exist $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, ρ, $\eta \in \mathcal{K}$ such that (4) and the implication

$$|x| \geq \rho(|r|) \ \Rightarrow\ \frac{\partial V}{\partial x} f(x, r) \leq -\eta(|x|) \tag{6}$$

holds for all $x \in \mathbb{R}^N$ and all $r \in \mathbb{R}^P$.

Proposition 5. System (1) is iISS (resp., ISS) if and only if it admits an iISS (resp., ISS) Lyapunov function.

The left hand side of (5) and the consequent of (6) is the time derivative of V along the trajectories $x(t)$ of system (1), i.e., $\dot{V} = (\partial V/\partial x) f$. Inequality (5) is often referred to as an dissipation inequality and its right-hand side $-\alpha(|x|) + \sigma(|r|)$ is called a supply rate. As introduced originally in Sontag and Wang [1995] and Angeli et al. [2000], the "implication" form (6) is used for defining the ISS Lyapunov function in the above, while the iISS Lyapunov function is defined in the "dissipation" form (5). For ISS, the two forms are qualitatively equivalent to each other. Indeed, the developments in Sontag and Wang [1995] and Angeli et al. [2000] yield the following property. It provides us with an explicit relationship between iISS and ISS in terms of supply rates.

Proposition 6. System (1) admits an ISS Lyapunov function if and only if there exist a continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}_+$, continuous functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ and $\alpha, \sigma \in \mathcal{K}$ such that (4) and (5) hold for all $x \in \mathbb{R}^N$ and all $r \in \mathbb{R}^P$, and

$$\lim_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} \sigma(s). \tag{7}$$

Furthermore, this equivalence remains valid even if (7) is replaced by $\alpha \in \mathcal{K}_\infty$.

Therefore, an ISS system is always iISS. The converse is not true. The function $\gamma \in \mathcal{K}$ in (3) is called the nonlinear gain function. From (6) it can be computed as $\gamma(s) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \rho(s)$ (Sontag and Wang [1995]). Obviously, the dissipation inequality (5) yields

$$\gamma(s) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1} \circ (\mathbf{Id} + \delta) \circ \sigma(s) \tag{8}$$

for $\delta \in \mathcal{K}_\infty$ in (3). It is stressed that if system (1) is not ISS, Proposition 6 implies that $\alpha^{-1} \circ (\mathbf{Id} + \delta) \circ \sigma(s)$ does not make sense. Thus, the nonlinear gain function γ is defined on $\mathbb{R}_+$ only if system (1) is ISS. If $\alpha \in \mathcal{K}$, $\gamma(s)$ is defined for $s \in \mathbb{R}_+$ satisfying $(\mathbf{Id} + \delta) \circ \sigma(s) < \lim_{\tau\to\infty} \alpha(\tau)$. The function $\mu \in \mathcal{K}$ in (2) is called the iISS gain function. From the dissipation inequality (5), we obtain $\mu = 2\sigma$ and $\chi = \underline{\alpha}$ as proved in Angeli et al. [2000].

When the nonlinear gain function is defined on $\mathbb{R}_+$ and linear, i.e., $\gamma(s) = ks$ for a constant $k \geq 0$, the nonlinear gain γ in (3) reduces to the operator gain of system (1). The function β provides the information about non-zero initial conditions. In the case where α and σ are restricted to quadratic functions, the dissipation inequality (5) is identical to the Hamilton Jacobi inequality in H^∞ control (van der Schaft [1999]).

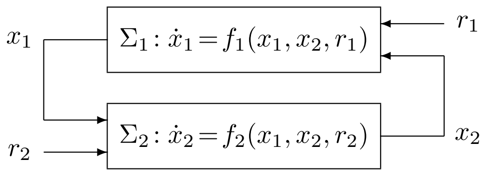


Fig. 1. Interconnected system Σ.

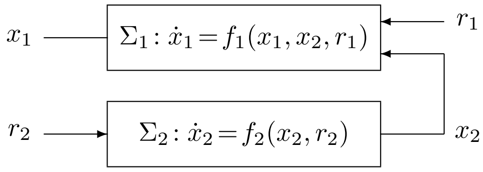


Fig. 2. Cascade system Σ.

Example 7. Consider a system Σ satisfying

$$\frac{\partial V}{\partial x} f(x, r) = -\alpha(|x|) + \sigma(|r|) \tag{9}$$

with some $\underline{\alpha} = \overline{\alpha} \in \mathcal{K}_\infty$ and some $\alpha, \sigma \in \mathcal{K}$. Then the system is iISS. Whether to be ISS or not depends on the upper limit of α and σ as follows:

$$\alpha(s) = s \quad, \ \sigma(s) = s \quad \Rightarrow \text{ISS}, \qquad \gamma = \alpha^{-1} \circ c\sigma \in \mathcal{K}$$

$$\alpha(s) = s \quad, \ \sigma(s) = \frac{s}{s+1} \Rightarrow \text{ISS}, \qquad \gamma = \alpha^{-1} \circ c\sigma \in \mathcal{K}$$

$$\alpha(s) = \frac{2s}{s+1}, \ \sigma(s) = \frac{s}{s+1} \Rightarrow \text{ISS}, \qquad \gamma = \alpha^{-1} \circ c\sigma \in \mathcal{K}$$

$$\alpha(s) = \frac{s}{s+1}, \ \sigma(s) = s \quad \Rightarrow \text{non-ISS}, \qquad \gamma(s) = \infty, s \geq 1$$

$$\alpha(s) = \frac{s}{s+1}, \ \sigma(s) = \frac{2s}{s+1} \Rightarrow \text{non-ISS}, \qquad \gamma(s) = \infty, s \geq 1$$

Here, we used $c \in (1, 2]$ for $\delta(s) = (c-1)s$ in (8).

3. OBSTACLES AND CLUES IN DEALING WITH INTERCONNECTIONS

3.1 Incompatibility of signal spaces

According to Definition 2, an ISS system is an $L^\infty \to L^\infty$ operator. Consider the interconnected system shown in Fig. 1. For each $i = 1, 2$, suppose that subsystem Σ_i is ISS with respect to input $x_{3-i} \in \mathbb{R}^{N_{3-i}}$. Let γ_i denote the nonlinear gain function of Σ_i with respect to input x_{3-i}. Then the interconnection is a loop consisting of two $L^\infty \to L^\infty$ operators and the open loop defined as the cascade of Σ_1 and Σ_2 is again an $L^\infty \to L^\infty$ operator. Thus, if the nonlinear gain of the open loop operator is strictly smaller than the identity map. i.e., if there exists ε_1, $\varepsilon_2 \in \mathcal{K}_\infty$ such that

$$(\mathbf{Id} + \varepsilon_1) \circ \gamma_1 \circ (\mathbf{Id} + \varepsilon_2) \circ \gamma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+, \tag{10}$$

then the signals $x_1(t)$ and $x_2(t)$ connecting the two subsystems must converge to zero. This is basically the argument of contraction mappings on which the iISS small-gain theorem is based (Jiang et al. [1994], Teel [1996], Sontag and Wang [1995]). Inequality (10) is often referred to as the small-gain condition. On the other hand, Definition 1 means that an iISS system looks like an $L^p \to L^\infty$ operator with $p < \infty$. Consider Fig. 1 again. Suppose that both Σ_1 and Σ_2 are iISS. If Σ_1 is not ISS, then it is not an $L^\infty \to L^\infty$ operators, but an $L^p \to L^\infty$ operator with $p < \infty$. The system Σ_2 is either $L^p \to L^\infty$ or $L^\infty \to L^\infty$, so that x_2 generated by Σ_2 is a function in L^∞ space. However, the input space of Σ_1 is L^p with $p < \infty$. In general, the L^p-norm has nothing to do with L^∞-norm. This incompatibility of signal spaces prevents us from applying the standard argument of contraction operators in the presence of an iISS subsystem which is not ISS.

3.2 Cascade of iISS is not always iISS

It is well known, and easy to prove, that the cascade of two ISS systems is again ISS. Indeed, two systems are $L^\infty \to L^\infty$ operators in Fig.2, and in particular, the nonlinear gain of the driving system Σ_2 with respect to the input x_1 in the cascade is the zero function $\gamma_2 = 0$. Obviously, condition (10) is met. When the cascade is allowed to contain iISS systems, an $L^p \to L^\infty$ operator is driven by another $L^p \to L^\infty$, and there is a mismatch between the signal spaces of the connecting channel. A cascade of two iISS system is is not always iISS. It is demonstrated in Arcak et al. [2002] that the solutions $x(t) = [x_1, x_2]^T \in \mathbb{R}^2$ of the cascade system

$$\dot{x}_1 = -\text{sgn}(x_1) \min\{1, |x_1|\} + x_1 x_2, \quad \dot{x}_2 = -x_2^3 \tag{11}$$

with $x_1(0) \geq 3$ and $x_2(0) = 1$ satisfy $x_1(t) \geq e^{(\sqrt{1+2t}-1)}$ which exhibits $x_1(t) \to \infty$ as $t \to \infty$. From Definition 2 it follows that the driven x_1-system is not ISS with respect to input x_2. It is verified that $V_1(x_1) = \ln(1 + x_1^2)$ is an iISS Lyapunov function of the x_1-system. The x_2-system is GAS, which is ISS with respect to the nil input. We have $\gamma_2 = 0$. However, γ_1 is not defined for (11).

3.3 Insufficiency of max-type Lyapunov functions

Suppose that $V_1(x_1) = |x_1|^2$ and $V_2(x_2) = |x_2|^2$ satisfy

$$\dot{V}_1 \leq -2|x_1|^2 + |x_2|^2 \tag{12}$$

$$\dot{V}_2 \leq -2|x_2|^2 + |x_1|^2 \tag{13}$$

along all possible trajectories $x_i(t)$ of subsystem Σ_i, $i = 1, 2$. Due to Proposition 6, the function $V_1(x_1)$ represents an ISS system Σ_1 with respect to input $x_2(t)$ and state $x_1(t)$, while $V_2(x_2)$ represents an ISS system Σ_2 with respect to input $x_1(t)$ and state $x_2(t)$. Thus, we have a feedback interconnection of two ISS systems in Fig.1. The nonlinear gain functions $\gamma_1(s) = \gamma_2(s) = \sqrt{c/2}s$ obtained from (12) and (13) with $c \in (1, 2)$ satisfy the small-gain condition (10) with $\varepsilon_i(s) = (\sqrt{2/c} - 1)s$. Hence, the interconnection represented by (12) and (13) is guaranteed to be GAS. Let $V(x) = \max\{V_1(x_1), V_2(x_2)\}$. Then $V_1(x_1) \geq V_2(x_2)$ implies $\dot{V} \leq -|x_1|^2$, and $V_1(x_1) \leq V_2(x_2)$ implies $\dot{V} \leq -|x_2|^2$, For all $x = [x_1^T, x_2^T]^T \in \mathbb{R}^N$ we arrive at $\dot{V} \leq -V$, so that $V(x)$ is a Lipschitz continuous Lyapunov function describing the GAS of the feedback interconnection. To link the small-gain condition (10) with the existence of a Lyapunov function $V(x)$ verifying GAS for dissipation inequalities more general than (12)-(13), consider

$$\underline{\alpha}(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}(|x_i|) \tag{14}$$

$$\dot{V}_1 \leq -(\mathbf{Id} + \bar{\delta}_1) \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\rho}^{-1} \circ \overline{\alpha}_1(|x_1|) + \sigma_1(|x_2|) \tag{15}$$

$$\dot{V}_2 \leq -(\mathbf{Id} + \bar{\delta}_2) \circ \sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\rho} \circ \overline{\alpha}_2(|x_2|) + \sigma_2(|x_1|) \tag{16}$$

Here, $\bar{\rho}$ is any continuously differentiable class $\mathcal{K}_\infty$ function satisfying $\bar{\rho}'(s) > 0$ for all $s \in (0, \infty)$. Suppose that $\sigma_i \in \mathcal{K}$, $\underline{\alpha}_i$, $\overline{\alpha}_i$, $\bar{\delta}_i \in \mathcal{K}_\infty$, $i = 1, 2$. Note that the pair (15)-(16) gives $\gamma_1 = \underline{\alpha}_1^{-1} \circ \bar{\rho} \circ \underline{\alpha}_2$ and $\gamma_2 = \underline{\alpha}_2^{-1} \circ \bar{\rho}^{-1} \circ \underline{\alpha}_1$ for $\delta_i = \bar{\delta}_i$, $i = 1, 2$. Hence, choosing $\delta_i(s) < \bar{\delta}_i(s)$ $\forall s \in (0, \infty)$ allows (10) to be satisfied for some ε_1, $\varepsilon_2 \in \mathcal{K}_\infty$. Let

$$V(x) = \max\{V_1(x_1), \bar{\rho}(V_2(x_2))\}. \tag{17}$$

Due to (14), $V_1(x_1) \geq \bar{\rho}(V_2(x_2))$ implies $\dot{V} \leq -\bar{\delta}_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\rho}^{-1}(V_1)$, and $V_1(x_1) \leq \bar{\rho}(V_2(x_2))$ implies $\dot{V} \leq \bar{\rho}'(V_2)(-\bar{\delta}_2 \circ \sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\rho}(V_2))$. Thus, $\dot{V} \leq -\eta(V)$ holds with a function $\eta \in \mathcal{K}$ for all $x \in \mathbb{R}^N$. Hence, under the assumption that Σ_1 and Σ_2 are ISS and satisfy (10), the function $V(x)$ in (17) is a Lipschitz continuous Lyapunov function establishing GAS of (15)-(16) (Jiang et al. [1996]). The right hand side of (12) and (15) is negative when $V_1(x_1)$ is large. The right hand side of (13) and (16) is negative if $V_2(x_2)$ is large. Thus, the time-derivative of $\max\{V_1(x_1), \bar{\rho}(V_2(x_2))\}$ is negative. Therefore, to make $V(x)$ in (17) be a Lyapunov function of the interconnection, property (7) is necessary for each Σ_1 and Σ_2. In other words, to establish GAS of the interconnection with (17), the dissipation inequality of Σ_i of each $i = 1, 2$ is required to guarantee ISS with respect input x_{3-i} and state x_i.

3.4 Energy conservation and dissipation

Suppose that $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$ satisfy

$$\dot{V}_1 \leq -2\frac{x_1^2}{1 + x_1^2} + x_2^2 \tag{18}$$

$$\dot{V}_2 \leq -2x_2^2 + \frac{x_1^2}{1 + x_1^2} \tag{19}$$

along the trajectories of $x_i(t)$ of Σ_i, $i = 1, 2$, in Fig. 1. Due to Proposition 6, Σ_1 is not guaranteed to be ISS but iISS. The presence of the non ISS system Σ_1 obviously implies that any function in the form of (17) cannot secure $\dot{V} \leq 0$ for all $x \in \mathbb{R}^N$ since the right hand side of (18) is positive whenever $\sqrt{|x_2|} \geq 2$. Now, rewrite (18)-(19) as

$$\dot{V}_1 \leq -2\frac{x_1^2}{1 + x_1^2} + x_2^2 = -q(x_1, x_2) - \frac{x_1^2}{1 + x_1^2} \tag{20}$$

$$\dot{V}_2 \leq q(x_1, x_2) - x_2^2. \tag{21}$$

The common term $q(x_1, x_2)$ in these two inequalities clarifies the structure of conservative energy interchange between V_1 and V_2, and the explicit energy dissipation of $-x_1^2/(1 + x_1^2)$ in Σ_1 and $-x_2^2$ in Σ_2. This structure implies the decrease of the quantity $V(x) = V_1(x_1) + V_2(x_2)$ over time, which proves GAS of (18)-(19), i.e., $\dot{V} \leq -x_1^2/(1 + x_1^2) - x_2^2$. While the nonlinear gain (8) of Σ_1 given by (18) explodes for the input $\sqrt{|x_2|} \geq 2$, the nonlinear gain of Σ_2 given by (19) is bounded. Nevertheless, the conservation plus dissipation exhibiting in (20)-(21) indicates that the blowup of Σ_1 is made up by the saturation of Σ_2. Interestingly, the formal application of (8) to (18) and (19) yields the pair

$$\gamma_1(s) = \sqrt{\frac{cs^2}{2 - cs^2}}, \quad \forall s \in [0, \sqrt{2/c}) \tag{22}$$

$$\gamma_2(s) = \sqrt{\frac{cs^2}{2(1 + s^2)}}, \quad \forall s \in \mathbb{R}_+ \tag{23}$$

that fulfills the small-gain condition (10) with $\varepsilon_i(s) = (\sqrt{2/c} - 1)s$, $i = 1, 2$, for $c \in (1, 2)$. Note that it is rare that the sum $V(x) = V_1(x_1) + V_2(x_2)$ leads us to GAS of interconnections. For instance, the pair (15)-(16) does not usually allow $V(x) = w_1 V_1(x_1) + w_2 V_2(x_2)$ to establish GAS with any constants w_1 and w_2. The limitation can be raised by using nonlinear transformations $W_1, W_2 \in \mathcal{K}_\infty$ as

$$V(x) = W_1(V_1(x_1)) + W_2(V_2(x_2)). \tag{24}$$

The example (18)-(19) suggests that the existence of transformations $W_1, W_2 \in \mathcal{K}_\infty$ by which the structure of conservation plus dissipation is revealed is expected to have a link to the small-gain condition (10) even when gains are not defined on the entire space $\mathbb{R}_+$.

4. INTERCONNECTED SYSTEMS IN IISS FRAMEWORK

Consider the interconnected system Σ described by

$$\Sigma : \begin{cases} \Sigma_1 : \dot{x}_1(t) = f_1(x_1(t), x_2(t), r_1(t)) \\ \Sigma_2 : \dot{x}_2(t) = f_2(x_1(t), x_2(t), r_2(t)), \end{cases} \tag{25}$$

where $x_i \in \mathbb{R}^{N_i}$, $r_i \in \mathbb{R}^{P_i}$ and $f_i : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{P_i} \to \mathbb{R}^{N_i}$, $i = 1, 2$. Defining $x = [x_1^T, x_2^T]^T \in \mathbb{R}^N$, $r = [r_1^T, r_2^T]^T \in \mathbb{R}^P$ and $f(x, r) = [f_1(x_1, x_2, r_1)^T, f_2(x_1, x_2, r_2)^T]^T \in \mathbb{R}^N$, the interconnected system (25) is identical to (1). Assume that the two subsystems Σ_i are iISS. Note that, for each i, the input of the subsystem Σ_i whose state vector is x_i is not only r_i, but also x_{3-i}. More precisely, the following is assumed.

Assumption 8. For each $i = 1, 2$, there exist a continuously differentiable function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, continuous functions $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$ and $\sigma_i \in \mathcal{K} \cup \{0\}$ and $\kappa_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|) \tag{26}$$

$$\frac{\partial V_i}{\partial x_i} f_i(x_1, x_2, r_i) \leq -\alpha_i(|x_i|) + \sigma_i(|x_{3-i}|) + \kappa_i(|r_i|) \tag{27}$$

hold for all $x_i \in \mathbb{R}^{N_i}$, $x_{3-i} \in \mathbb{R}^{N_{3-i}}$ and all $r_i \in \mathbb{R}^{P_i}$.

The pair (26)-(27) implies

$$\frac{\partial V_i}{\partial x_i} f_i(x_1, x_2, r_i) \leq \\ - \alpha_i(\overline{\alpha}_i^{-1}(V_i)) + \sigma_i(\underline{\alpha}_{3-i}^{-1}(V_{3-i})) + \kappa_i(|r_i|) \tag{28}$$

Thus, for brevity, this paper replaces Assumption 8 by the following.

Assumption 9. For each $i = 1, 2$, there exist a continuously differentiable function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, continuous functions $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$ and $\sigma_i \in \mathcal{K} \cup \{0\}$ and $\kappa_i \in \mathcal{K}_\infty$ such that (26) holds for all $x_i \in \mathbb{R}^{N_i}$, and that

$$\frac{\partial V_i}{\partial x_i} f_i(x_1, x_2, r_i) \leq -\alpha_i(V_i(x_i)) + \sigma_i(V_{3-i}(x_{3-i})) + \kappa_i(|r_i|) \tag{29}$$

holds for all $x_i \in \mathbb{R}^{N_i}$, $x_{3-i} \in \mathbb{R}^{N_{3-i}}$ and all $r_i \in \mathbb{R}^{P_i}$.

Remark 10. Due to (26), $V_i(x_i(t))$ is qualitatively identical to $|x_i(t)|$. The left hand side of (29) is the time-derivative of V_i. Hence, for arbitrary dimensions N_1 and N_2, system (25) defined with the state $x = [x_1^T, x_2^T]^T \in \mathbb{R}^N$ is condensed into the low-dimensional system (29) defined with the planar state $[V_1, V_2]^T \in \mathbb{R}_+^2$. In fact, the dissipation inequality of iISS type allows us to utilize comparison

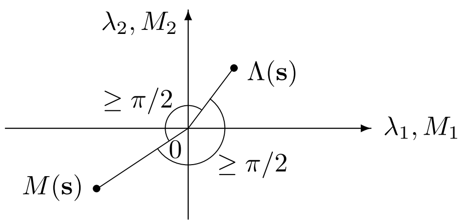
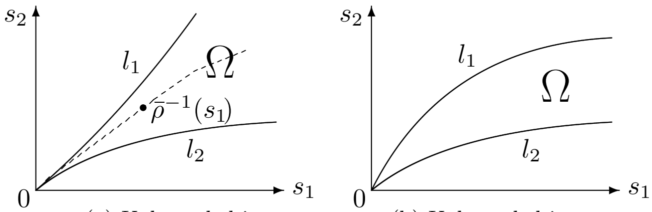


Fig. 3. Geometrical interpretation of (34).



(a) Unbounded in all directions.

(b) Unbounded in a single direction.

Fig. 4. Topological interpretation of (48).

principles in which the behavior of the interconnected iISS systems can be deduced from the planar positive system

$$\dot{V}_i = -\alpha_i(V_i) + \sigma_i(V_{3-i}) + \kappa_i(|r_i|), \ i = 1, 2$$

with $\bar{V}(0) = [V_1(0), V_2(0)]^T \in \mathbb{R}_+^2$ (see e.g. Lakshmikantham and Leela [1969], Smith [1995], Rüffer et al. [2010], Angeli and Astolfi [2007]).

5. CONSTRUCTION OF LYAPUNOV FUNCTIONS

5.1 Sum-type Lyapunov functions

Consider the function in (24) with continuously differentiable functions $W_1, W_2 \in \mathcal{K}_\infty$ whose derivatives are positive for all values except the origin. Equivalently, using continuous functions $\lambda_i = W_i' : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, we define

$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s) ds + \int_0^{V_2(x_2)} \lambda_2(s) ds. \tag{30}$$

$$\lambda_i(s) > 0, \quad s \in (0, \infty). \tag{31}$$

From (29) it follows that

$$\frac{dV}{dt}(x, r) \leq \Lambda(\bar{V}(x))^T S(\bar{V}(x), r) \tag{32}$$

holds along the solutions of system (25), where

$$\bar{V}(x) = [\, V_1(x_1) \ V_2(x_2) \,]^T$$
$$\Lambda(\mathbf{s}) = [\, \lambda_1(s_1) \ \lambda_2(s_2) \,]^T, \quad \mathbf{s} = [s_1, s_2]^T \in \mathbb{R}_+^2$$
$$S(\mathbf{s}, r) = \begin{bmatrix} -\alpha_1(s_1) + \sigma_1(s_2) + \kappa_1(|r_1|) \\ -\alpha_2(s_2) + \sigma_2(s_1) + \kappa_2(|r_2|) \end{bmatrix}.$$

Proposition 5 allows us to recast iISS of system (25) as the problem of finding continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, that satisfy (31) and achieve

$$\Lambda(\mathbf{s})^T S(\mathbf{s}, r) \leq -\alpha(|\mathbf{s}|) + \sigma(|r|), \ \forall \mathbf{s} \in \mathbb{R}_+^2, \ r \in \mathbb{R}^P \tag{33}$$

for some $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$. Given $c_i > 1$, $i = 1, 2$, define an operator $M : \mathbb{R}_+^2 \to \mathbb{R}^2$ as

$$M(\mathbf{s}) = \begin{bmatrix} -c_1^{-1}\alpha_1(s_1) + \sigma_1(s_2) \\ -c_2^{-1}\alpha_2(s_2) + \sigma_2(s_1) \end{bmatrix} = \begin{bmatrix} M_1(\mathbf{s}) \\ M_2(\mathbf{s}) \end{bmatrix}.$$

The following is straightforward:

Proposition 11. Suppose that continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfy (31). If there exist $c_i > 1$, $i = 1, 2$, such that

$$\Lambda(\mathbf{s})^T M(\mathbf{s}) \leq 0, \quad \forall \mathbf{s} \in \mathbb{R}_+^2 \tag{34}$$

holds, there exists $\alpha \in \mathcal{P}$ achieving (33) with $r = 0$.

The technique of changing supply rates in Sontag and Teel [1995] yields the following (Ito et al. [2012]):

Proposition 12. Suppose that continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfy (31) and the implication

$$\{\alpha_i \in \mathcal{K} \setminus \mathcal{K}_\infty \ \Rightarrow \ \limsup_{s \to \infty} \lambda_i(s) < \infty\}, \ i = 1, 2. \tag{35}$$

If there exist $c_i > 1$, $i = 1, 2$, such that (34) holds, there exist $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ achieving (33). Moreover, the function α is of class $\mathcal{K}_\infty$ if $\lim_{s \to \infty} \alpha_i(s) = \infty$ and $\liminf_{s \to \infty} \lambda_i(s) > 0$ hold for $i = 1, 2$ additionally.

Propositions 11 and 12 recast the construction of a sum-type Lyapunov function V of the form (30) as the problem of finding Λ fulfilling (34) in both of GAS and iISS cases. Figure 3 illustrates (34). The angles enclosed by two vectors $\Lambda(\mathbf{s})$ and $M(\mathbf{s})$ are greater than or equal to $\pi/2$ if and only if (34) holds. For (18)-(19), a solution to (34) is $\lambda_i(s_i) = 1$, $i = 1, 2$.

5.2 iISS small-gain condition

To present a solution to the central problem (34), we define an operator $\alpha_i^{\ominus}$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as

$$\alpha_i^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \geq \alpha_i(v)\}. \tag{36}$$

Thus, we have $\alpha_i^{\ominus}(s) = \infty$ for $s \geq \lim_{\tau \to \infty} \alpha_i(\tau)$, and $\alpha_i^{\ominus}(s) = \alpha_i^{-1}(s)$ elsewhere. For a class $\mathcal{K}$ function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, this paper uses the extension ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ defined as

$$\omega(s) := \sup_{v \in \{w \in \mathbb{R}_+ \,:\, w \leq s\}} \omega(v).$$

The reader may refer to Ito et al. [2013b] for the benefit of these extended operators. We begin with system (25) in the absence of the disturbance r.

Theorem 13. If there exist $\bar{c}_i > 1$, $i = 1, 2$, such that

$$\alpha_1^{\ominus} \circ \bar{c}_1 \sigma_1 \circ \alpha_2^{\ominus} \circ \bar{c}_2 \sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}^+ \tag{37}$$

holds, the function $V : \mathbb{R}^n \to \mathbb{R}_+$ given in (30) with

$$\lambda_i(s) = \left[\frac{1}{\tau_i}\alpha_i(s)\right]^{\varphi} [\sigma_{3-i}(s)]^{\varphi+1}, \ i = 1, 2 \tag{38}$$

is a Lyapunov function establishing GAS of $x = 0$ of system (25) with $r(t) \equiv 0$, where $\tau_i > 0$ and $\varphi \geq 0$ are any real numbers satisfying

$$1 < \tau_i < \bar{c}_i, \quad \left(\frac{\tau_i}{\bar{c}_i}\right)^{\varphi+1} \leq \tau_i - 1, \quad i = 1, 2. \tag{39}$$

Proof. Inequality (31) is satisfied by (38). Property (37) and the definition of $\alpha_i^{\ominus}$ imply $\sigma_{3-i} \circ \alpha_i^{\ominus} \circ \bar{c}_i \sigma_i \in \mathcal{J}$ for $i = 1, 2$. From this property, (38) and (39) it follows that $\lambda_i \circ \alpha_i^{\ominus} \circ \tau_i \sigma_i \in \mathcal{J}$. This allows us to consider the two separate cases $\alpha_i(s_i) \geq \tau_i \sigma_i(s_{3-i})$ and $\alpha_i(s_i) < \tau_i \sigma_i(s_{3-i})$ for each $i = 1, 2$ to verify (34) with $1 < c_i \leq \bar{c}_i$, $i = 1, 2$, as in Ito [2006]. Finally, Proposition 11 and (32) with $r = 0$ prove the GAS. $\square$

The next theorem addresses robustness of system (25) with respect to the disturbance r.

Theorem 14. Suppose that

$$\left\{\lim_{s\to\infty}\alpha_i(s)=\infty \ \vee \ \lim_{s\to\infty}\sigma_{3-i}(s)<\infty\right\},\ i=1,2 \quad (40)$$

holds. If there exist $\bar{c}_i>1$, $i=1,2$, such that (37) holds, the function $V:\mathbb{R}^n\to\mathbb{R}_+$ given in (30) with (38) is an iISS Lyapunov function of system (25), where $\tau_i>0$ and $\varphi\geq 0$ are any real numbers satisfying (39). Moreover, if $\lim_{s\to\infty}\alpha_i(s)=\infty$ holds for $i=1,2$, the function V is an ISS Lyapunov function of system (25).

Proof. Assumption (40) ensures that λ_i in (38) satisfies the implication (35) for $i=1.2$. Properties (31) and (34) are verified as in the proof of Theorem 13. Propositions 5, 6, and 12 together with (32) complete the proof. □

The necessity of (40) can be addressed by the next proposition (Ito et al. [2013b]).

Proposition 15. Assume that $\alpha_i\in\mathcal{K}$ and $\sigma_i\in\mathcal{K}\cup\{0\}$ and $\kappa_i\in\mathcal{K}_\infty$ are given for $i=1,2$. Suppose that there exists an iISS Lyapunov function in the form of (30) for all Σ satisfying Assumption 9. Then property (40) holds.

Example 16. Consider (29) with (26) given by

$$\alpha_1(s)=3s^2,\quad \sigma_1(s)=s^2,\quad r_1(t)=0 \quad (41)$$

$$\alpha_2(s)=\frac{3s^2}{1+s^2},\quad \sigma_2(s)=\frac{s^2}{1+s^2},\quad r_2(t)=0 \quad (42)$$

$$\underline{\alpha}_i=\overline{\alpha}_i,\quad i=1,2. \quad (43)$$

Since (37) is satisfied with $\bar{c}_i=3$, $i=1,2$, by virtue of Theorem 13, any system Σ in (25) fulfilling (29) with (41)-(42) is GAS. To construct a Lyapunov function, pick $\psi=0$ and $\tau_i=2$, $i=1,2$, achieving (39). Equation (38) gives

$$\lambda_1(s)=\frac{s^2}{1+s^2},\quad \lambda_2(s)=s^2.$$

It is verified that V in (30) satisfies

$$\Lambda(\bar{V})^TS(\bar{V},r)\leq\sum_{i=1}^{2}\left(-\frac{2V_i^4}{1+V_i^2}+\frac{V_{3-i}^4}{1+V_{3-i}^2}\right)\leq-\sum_{i=1}^{2}\frac{V_i^4}{1+V_i^2}.$$

The first inequality is obtained by considering the two separate cases $V_1\geq V_2$ and $V_1<V_2$. Hence, the function V in (30) is a Lyapunov function establishing GAS of system (25) with $r(t)\equiv 0$. Note that the pair (41)-(42) implies that Σ_1 and Σ_2 are ISS. Indeed, the small-gain condition (37) implicitly requires both systems to be ISS when (40) is not satisfied. This example violates (40) for $i=2$.

Example 17. Consider the following functions with $h\geq 1$ for the dissipation inequality (29) and (26):

$$\alpha_1(s)=\left(\frac{s^2}{1+s^2}\right)^h,\ \sigma_1(s)=\left(\frac{2s^2}{1+s^2}\right)^h,\ \kappa_1(s)=s \quad (44)$$

$$\alpha_2(s)=\frac{5s^2}{1+s^2},\quad \sigma_2(s)=\frac{2s^2}{6+5s^2},\quad \kappa_2(s)=s \quad (45)$$

$$\underline{\alpha}_i=\overline{\alpha}_i,\quad i=1,2. \quad (46)$$

The second system Σ_2 given by (45) is ISS, while (44) guarantees Σ_1 to be only iISS. The above functions in (44)-(45) fulfill (40). The choice $\bar{c}_i=2.5$, $i=1,2$, satisfies (37). Using $\psi=0$ and $\tau_i=2$, $i=1,2$, satisfying (39) we obtain

$$\lambda_1(s)=\frac{2s^2}{6+5s^2},\quad \lambda_2(s)=\left(\frac{2s^2}{1+s^2}\right)^h$$

from (38). Defining V as in (30) we have

$$\begin{aligned}\Lambda^TS\leq&-\frac{2V_1^2}{6+5V_1^2}\left[\left(\frac{V_1^2}{1+V_1^2}\right)^h-\left(\frac{2V_1^2}{6+5V_1^2}\right)^h\right]+\left(\frac{2V_2^2}{1+V_2^2}\right)^{h+1}\\&+\frac{2}{5}|r_1|-3\cdot 2^h\left(\frac{V_2^2}{1+V_2^2}\right)^h+\left(\frac{2V_1^2}{6+5V_1^2}\right)^{h+1}+2^h|r_2|\\\leq&-\frac{1}{3}\left[1-2\left(\frac{2}{5}\right)^h\right]\left(\frac{V_1^2}{1+V_1^2}\right)^{h+1}-2^h\left(\frac{V_2^2}{1+V_2^2}\right)^{h+1}\\&+\frac{2}{5}|r_1|+2^h|r_2|.\end{aligned}$$

The second inequality is due to the two separate cases $2V_1^2/(6+5V_1^2)\lesseqqgtr 2V_2^2/(1+V_2^2)$. Thus, iISS of system (25) is established by V in (30) as guaranteed by Theorem 14.

5.3 Alternative condition

The following equivalence can be proved.

Theorem 18. Let $\hat{c}_i>1$, $i=1,2$. The property

$$\alpha_1^{\ominus}\circ\hat{c}_1\sigma_1\circ\alpha_2^{\ominus}\circ\hat{c}_2\sigma_2(s)<s,\quad \forall s\in\mathbb{R}^+\setminus\{0\} \quad (47)$$

holds if and only if

$$M(\mathbf{s})\not\geq 0,\quad \forall \mathbf{s}\in\mathbb{R}_+^2\setminus\{0\} \quad (48)$$

is satisfied with $c_i=\hat{c}_i$, $i=1,2$.

The existence of $\hat{c}_i>1$, $i=1,2$, satisfying (47) implies and is implied by the existence of $\bar{c}_i>1$, $i=1,2$, satisfying (37). Indeed, property (47) guarantees (37) with $\bar{c}_i=\hat{c}_i$. The converse holds true with $\hat{c}_i=\bar{c}_i/2$. Hence, the obtuse angle problem (34) illustrated by Fig.3 is recast as (48). The condition (48) is given a topological interpretation in Fig.4. Property (48) ensures that the open set

$$\Omega=\{s=[s_1,s_2]^T\in\mathbb{R}_+^2:M(\mathbf{s})\ll 0\} \quad (49)$$

divides $\mathbb{R}_+^2\setminus\{0\}$ into two disjoint sets. Under the map $M:\mathbb{R}_+^2\to\mathbb{R}^2$, the set Ω is the preimage of the open negative orthant in $\mathbb{R}^2$, i.e., the third quadrant in Fig.3. The boundary of Ω is given by the two curves l_i: $\alpha_i(s_i)=c_i\sigma_{i,3-i}(s_{3-i})$, $i=1,2$. The curve l_i is identical with the s_{3-i}-axis if $\sigma_{i,3-i}=0$. The unboundedness of Ω in the s_{3-k} direction is equivalent to the ISS property of Σ_k since the unboundedness is equivalent to $\alpha_k(\infty)\geq c_k\sigma_{k,k-3}(\infty)$. Figure 4 (b) illustrates the case where only Σ_2 is ISS and Ω is unbounded only in the s_1 direction. The boundedness of Ω in the s_2 direction allows Σ_1 to be non-ISS.

It is interesting to notice that based on Fig.4 (a), Jiang et al. [1996] constructed the max-type Lyapunov function (17). We can choose the strictly increasing function $\bar{\rho}$ so that the curve $s_1=\bar{\rho}(s_2)$ is a subset of Ω connecting the origin and (∞,∞) in Fig. 4 (a), i.e.,

$$M([s_1,\bar{\rho}^{-1}(s_1)]^T)\ll 0,\quad \forall s_1\in\mathbb{R}_+\setminus\{0\}\ . \quad (50)$$

The existence of such a curve is guaranteed if (48) is satisfied, provided that the two subsystems are ISS (Jiang et al. [1996], Dashkovskiy et al. [2010]). Here, to ensure that V in (17) is a radially unbounded function defined on $\mathbb{R}^N$, the function $\bar{\rho}$ needs to be defined on $\mathbb{R}_+$ and unbounded. This implies that the set Ω must be unbounded in both s_1 and s_1 directions. Thus, the max-type Lyapunov function (17) requires both subsystems to be ISS (Ito et al. [2012]).

5.4 Left and right eigenvectors

Suppose that M is linear, and let $\breve{M}$ denote its matrix representation. Then (34) holds if $(\Lambda(\mathbf{s})\breve{M})^T < 0$, $\forall \mathbf{s} \in \mathbb{R}_+^2 \setminus \{0\}$. The argument of the Perron-Frobenius theorem yields that there exits a constant vector $\Lambda \in (0,\infty)^2$ if and only if the largest eigenvalue of $\breve{M}$ is negative. A constant vector Λ is obtained as a corresponding left eigenvector of the matrix $\breve{M}$. Next, consider a linear function $\bar{\rho}$ and let $\breve{\rho} > 0$ denote its coefficient. Then (50) holds if and only if $\breve{M}[1, 1/\breve{\rho}^{-1}]^T < 0$ holds for the constant $\breve{\rho} > 0$. The Perron-Frobenius theorem again yields that there exits such a constant $\breve{\rho} > 0$ if and only if the largest eigenvalue of $\breve{M}$ is negative. A choice of $[1, 1/\breve{\rho}^{-1}]^T$ is a corresponding right eigenvector of $\breve{M}$. Thus, if M is linear, a solution to the obtuse angle problem (34) in Fig.3 is a left eigenvector, while a solution to the separation problem (50) in Fig.4 (a) is a right eigenvector associated with M (Dashkovskiy et al. [2011]). The linearity of M implies ISS of the two subsystems. Notice that the left eigenvector remains a solution to (34) even when $M(s) = \breve{M}[m_1(s_1), m_2(s_2)]^T$ holds for some m_1, $m_2 \in \mathcal{P}$. Thus, the left eigenvector approach (34) is effective even if subsystems are not ISS.

5.5 Cascades

The interconnection (25) becomes a cascade system when $\sigma_2 = 0$. According to Theorem 13, under Assumption 9, an interconnection of two iISS subsystems is always GAS. However, it is emphasized that $\alpha_1, \alpha_2 \in \mathcal{K}$ is assumed. The example (11) does not admit $\alpha_1 \in \mathcal{K}$. Property $\alpha_1 \in \mathcal{P}\setminus\mathcal{K}$ disqualifies Theorem 13. In the presence of disturbances, we can invoke Theorem 14. Thus, a cascade is always iISS if either $\alpha_2 \in \mathcal{K}_\infty$ or $\sigma_1 \notin \mathcal{K}_\infty$ holds as long as $\alpha_1, \alpha_2 \in \mathcal{K}$. It is possible to remove the constraint $\alpha_1, \alpha_2 \in \mathcal{K}$ with the help of growth rate restrictions on functions describing interconnection near the origin (e.g. Panteley and Loría [1998, 2001], Arcak et al. [2002], Chaillet and Angelli [2008], Ito [2010]).

6. TIME DELAYS IN INTERCONNECTIONS

The speed and direction of trajectories (i.e., vector fields) of systems involving time-delays are determined not only by the current state and input, but also by the past state and input. Thus, evolutions of delay systems reside in function spaces. To treat this infinite dimensional character, the concept of Lyapunov functions can be extended to Lyapunov-Krasovskii functionals, and we can make use of dissipativity. It is not at all difficult to restate everything in the previous sections by replacing Euclidean spaces by function spaces. For instance, a functional V can be constructed as in (30) for interconnections of subsystems described by functionals V_1 and V_2. However, this formal extension is insufficient for the treatment of delay systems. Let the functional V_i represent the energy of Σ_i. This functional V_i contains information about internal variables of Σ_i. Thus, the effect of internal delays are built in V_i. The dissipation in the form of (29) requires the effect of Σ_{3-i} on Σ_i to be expressed only through V_{3-i}. Hence, the characterization (29) prevents us from dealing with delays arising in communication channels. Moreover, if V_{3-i} were used for describing a discrete delay in the channel connecting Σ_{3-i} to Σ_i in (29), V_{3-i} is not entitled to be an appropriate energy functional of Σ_{3-i}. Therefore, to deal with delay systems, we need to modify (29). Let $\mathcal{C}_{N_i}$ denote the space of continuous functions mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^{N_i}$, where $\Delta > 0$ is the maximum involved delay. We modify Assumption 9 as follows:

Assumption 19. For $i = 1, 2$, there exist Locally Lipschitz functionals $V_i : \mathcal{C}_{N_i} \to \mathbb{R}_+$, and $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$, $\kappa_i \in \mathcal{K}_\infty$, and continuous functions $\|\cdot\|_{a,i} : \mathcal{C}_{N_i} \to \mathbb{R}_+$, and integers $h, h_d \geq 0$ such that

$$\underline{\gamma}_{a,i}(|\phi_i(0)|) \leq \|\phi_i\|_{a,i} \leq \overline{\gamma}_{a,i}(\|\phi_i\|_\infty) \tag{51}$$

$$\underline{\alpha}_i(\|\phi_i\|_{a,i}) \leq V_i(\phi_i) \leq \overline{\alpha}_i(\|\phi_i\|_{a,i}), \tag{52}$$

$$\begin{aligned} D^+V_i(\phi_i, \phi_{3-i}, r_i) \leq -\alpha_i(V_i(\phi_i)) + \sigma_{i,0}(V_i(\phi_{3-i})) \\ + \sum_{j=1}^{h} \sigma_{i,j}\left(|\phi_{3-i}(-\Delta_j|\right) + \sum_{j=h+1}^{h+h_d} \int_{-\Delta_j}^{0} \sigma_{i,j}\left(|\phi_{3-i}(\tau)|\right) d\tau \\ + \kappa_i(|r_i|) \end{aligned} \tag{53}$$

hold for all $\phi = [\phi_1^T, \phi_2^T]^T \in \mathcal{C}_N$ and all $r_i \in \mathbb{R}^{P_i}$, where $\Delta_j \in (0, \Delta]$ for $j = 0, 1, \ldots, h + h_d$.

Here, D^+V_i associated with Σ_i is the derivative used in Pepe and Jiang [2006]. The third term in (53) represents discrete delays in the connecting channel, while the fourth term represents distributed delays. To deal with these new terms, replace (30) by

$$V(\phi) = \sum_{i=1}^{2} \left\{ \int_0^{V_i(\phi_i)} \lambda_i(s) ds + \sum_{j=1}^{h+h_d} X_i(\phi_{3-i}) \right\}. \tag{54}$$

For details of X_i constructed from λ_i, see Ito et al. [2010]. Defining $\sigma_i \in \mathcal{K}$ by $\hat{\sigma}_{i,j} = \sigma_{i,j} \circ \gamma_{a,3-i}^{-1} \circ \underline{\alpha}_{3-i}^{-1}$ and

$$\begin{aligned} \sigma_i(s) = (1+\mu_i) \sum_{j=0}^{h+h_d} \mathrm{sgn}(\sigma_{i,j}(1)) \max\Big\{ \sigma_{i,0}(s), \hat{\sigma}_{i,j}(s) \\ \max_{j=1,\ldots,h} \hat{\sigma}_{i,j}(s), \max_{j=h+1,\ldots,h+h_d} \Delta_j \hat{\sigma}_{i,j}(s) \Big\} \end{aligned} \tag{55}$$

for $i = 1, 2$, Theorems 13 and 14 hold true for the above functional V for any $\mu_i > 0$. iISS and ISS of time-delay systems are the same as (2) and (3), respectively, except that $|x(0)|$ is replaced by $\|\xi\|_\infty$ for the initial condition $\xi \in \mathcal{C}_N$. iISS and ISS Lyapunov-Krasovskii functionals are defined in Pepe and Jiang [2006], Ito et al. [2010].

7. CONCLUDING REMARKS

This paper has attempted to provide the reader with an outline of a recent development on the utilization of iISS in constructing Lyapunov functions for interconnected systems. This paper first exemplified the usefulness of the iISS framework for finite-dimensional systems. Then, using delay systems, we illustrated a case where dissipation inequalities need to be modified, and explained the idea of reducing the modified problems into the same mathematical framework through a judicious selection of Lyapunov-Krasovskii functionals. Although it is not explained in this paper, it is possible to address the necessity of the iISS small-gain condition by considering sets of subsystems satisfying Assumption 9 as in Ito and Jiang [2009].

Construction of composite Lyapunov functions is not the only way to tackle interconnections of iISS systems. For example, we can take the monotone systems approach as demonstrated in Angeli and Astolfi [2007] in the absence of disturbances. It is also possible to exploit the ISS small-gain condition for non-ISS subsystems by assuming that the behavior of the subsystems is ISS after a transient period and that a trajectory estimate of the interconnection during that period is available in a desired manner as discussed in Karafyllis and Jiang [2012] and Ito et al. [2013a].

Because this paper is essentially a collection of published materials and some updates, the reader can refer to the appropriate references for detailed statements of results.

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REFERENCES

D. Angeli and A. Astolfi. A tight small gain theorem for not necessarily ISS systems. *Syst. Control Lett.*, 56:87–91, 2007.

D. Angeli and E.D. Sontag. Monotone control systems. *IEEE Trans. Automat. Contr.*, 48:1684–1698, 2003.

D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Contr.*, 45:1082–1097, 2000.

M. Arcak, D. Angeli, and E. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. Control Optim.*, 40:1888–1904, 2002.

A. Chaillet and D. Angelli. Integral input to state stable systems in cascade. *Syst. Control Lett.*, 57:519–527, 2008.

S. Dashkovskiy, B. Rüffer, and F. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48:4089–4118, 2010.

S. Dashkovskiy, H. Ito, and F. Wirth. On a small-gain theorem for ISS networks in dissipative Lyapunov form. 17:357–369, 2011.

G. Enciso and E.D. Sontag. Global attractivity, I/O monotone small-gain theorems and biological delay systems. *Discrete and Continuous Dynamical Systems*, 14:549–578, 2006.

D.J. Hill. A generalization of the small-gain theorem for nonlinear feedback systems. *Automatica*, 27:1043–1045, 1991.

D.J. Hill and P.J. Moylan. Stability results for nonlinear feedback systems. *Automatica*, 13:377–382, 1977.

H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Automat. Contr.*, 51:1626–1643, 2006.

H. Ito. A Lyapunov approach to cascade interconnection of integral input-to-state stable systems. *IEEE Trans. Automat. Contr.*, 55:702–708, 2010.

H. Ito and Z.P. Jiang. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective. *IEEE Trans. Automat. Contr.*, 54:2389–2404, 2009.

H. Ito, P. Pepe, and Z.P. Jiang. A small-gain condition for iISS of interconnected retarded systems based on Lyapunov-Krasovskii functionals. *Automatica*, 46:1646–1656, 2010.

H. Ito, S. Dashkovskiy, and F. Wirth. Capability and limitation of max- and sum-type construction of Lyapunov functions for networks of iISS systems. *Automatica*, 48: 1197–1204, 2012.

H. Ito, R.A. Freeman, and A. Chaillet. Revisiting the iISS small-gain theorem through transient plus ISS small-gain regulation. *Asian J. Control*, 15:11–19, 2013a.

H. Ito, Z.P. Jiang, S. Dashkovskiy, and B. Rüffer. Robust stability of networks of iISS systems: construction of sum-type Lyapunov functions. *IEEE Trans. Automat. Contr.*, 58:1192–1207, 2013b.

Z.P. Jiang, A.R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Contr. Signals and Syst.*, 7:95–120, 1994.

Z.P. Jiang, I. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32:1211–1215, 1996.

I. Karafyllis and Z.P. Jiang. A new small-gain theorem with an application to the stabilization of the chemostat. *Int. J. Robust and Nonlinear Contr*, 22:1602–1630, 2012.

V. Lakshmikantham and S. Leela. *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations.* Academic, New York, 1969.

I.M.Y. Mareels and D.J. Hill. Monotone stability of nonlinear feedback systems. *J. Math. Systems, Estim. Control*, 2:275–291, 1992.

E. Panteley and A. Loría. On global uniform asymptotic stability of nonlinear time-varying systems in cascade. *Syst. Control Lett.*, 33:131–138, 1998.

E. Panteley and A. Loría. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. *Automatica*, 37:453–460, 2001.

P. Pepe and Z.P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Contr. Letters*, 55:1006–1014, 2006.

B.S. Rüffer, C.M. Kellett, and S.R. Weller. Connection between cooperative positive systems and integral input-to-state stability of large-scale systems. *Automatica*, 46: 1019–1027, 2010.

H.L. Smith. *Monotone dynamical systems.* American Mathematical Society, Providence, 1995.

E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Contr.*, 34:435–443, 1989.

E.D. Sontag. Comments on integral variants of ISS. *Systems & Contr. Letters*, 34:93–100, 1998.

E.D. Sontag and A.R. Teel. Changing supply functions in input/state stable systems. *IEEE Trans. Automat. Contr.*, 40:1476–1478, 1995.

E.D. Sontag and Y. Wang. On characterizations of input-to-state stability property. *Systems & Contr. Letters*, 24:351–359, 1995.

A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. Automat. Contr.*, 41:1256–1270, 1996.

A. van der Schaft. *$\mathcal{L}_2$-gain and passivity techniques in nonlinear control.* Springer, New York, 1999.

G. Zames. On the input-output stability of time-varying nonlinear feedback systems. part i: conditions usingconcepts of loop gain, conicity, and positivity. *IEEE Trans. Automat. Contr.*, 11:228–238, 1966.